Détection de sources astrophysiques dans les données du spectrographe intégral de champ MUSE

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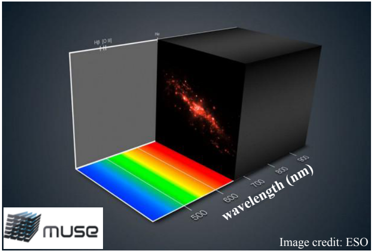
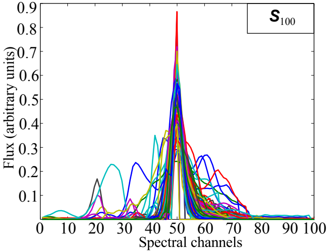
Porquerolles, 15 mai 2014

Hyperspectral Imaging with MUSE: detection of galaxies and emission lines

The spectrograph MUSE delivers data cube of 300×300 *pixels at* 3600 *wavelength channels. Lyman-*α *emitters are very distant and faint; contain essentially one emission line, whose profile varies with the considered object.*



MUSE (Multi Unit Spectroscopic Explorer) data cube



100 possible alternatives under $\mathcal{H}_1$

Detect one target signature $\boldsymbol{s}_i \in \mathbb{R}^{N\times 1}$ which belongs to a large known library $\boldsymbol{S} = [\boldsymbol{s}_1, \ldots, \boldsymbol{s}_L]$ ($\boldsymbol{S} \in \mathbb{R}^{N\times L}$, with $L \gg N$).

The amplitude and the alternative $\boldsymbol{s}_\ell$ activated under $\mathcal{H}_1$ are unknown.

Strategy for dimension reduction: Average vs Minimax performance

Standard approach when having a very large library is to test in subspaces of reduced dimensions (*e.g.*: sample mean, SVD, K-SVD, etc).

+ less computationally complex with similar average performances.

— might induce drop of performance for some targets.

Proposed approach

Learn a reduced size library with the objective of minimizing the maximum power loss.

Exact model and associated GLRT

$$\begin{cases} \mathcal{H}_0 & : \boldsymbol{x} = \boldsymbol{n}, \qquad \boldsymbol{n} \sim \mathcal{N}(\mathbf{0}, \mathbf{I}) \\ \mathcal{H}_1 & : \boldsymbol{x} = \boldsymbol{S}\boldsymbol{\alpha} + \boldsymbol{n}, \quad ||\boldsymbol{\alpha}||_0 = 1 \end{cases}$$

$\boldsymbol{n} \in \mathbb{R}^{N\times 1}$: gaussian noise (covariance matrix is known and equal under $\mathcal{H}_0$ and $\mathcal{H}_1$),
$\boldsymbol{S} \in \mathbb{R}^{N\times L} = [\boldsymbol{s}_1, \boldsymbol{s}_2, \ldots, \boldsymbol{s}_L]$ with $||\boldsymbol{s}_i||_2^2 = 1\ \forall i$, $L \gg N$, and $\boldsymbol{\alpha}$: unknown vector.

GLR test using $\boldsymbol{S}$ under constraint $||\boldsymbol{\alpha}||_0 = 1$:

$$\max_{\boldsymbol{\alpha}:||\boldsymbol{\alpha}||_0=1} \frac{p(\boldsymbol{x}|\boldsymbol{S}\boldsymbol{\alpha})}{p(\boldsymbol{x}|\mathbf{0})} \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \gamma'$$

Corresponding GLRT: *Max* test over *S*

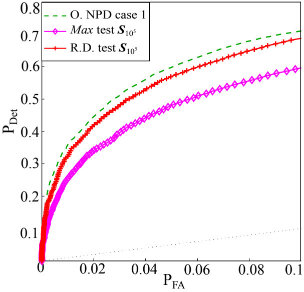
$$\Rightarrow T_{\text{Max}} : \max_{i=1,\ldots,L} |\boldsymbol{s}_i^\top \boldsymbol{x}| \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \gamma\,.$$

Issues when testing with a reduced size library

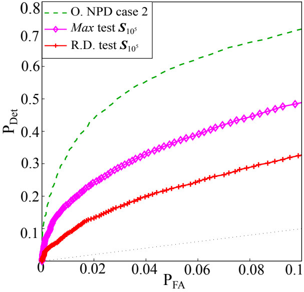
Assuming in all cases that $\boldsymbol{s}_\ell$ is the active alternative under $\mathcal{H}_1$:

- Oracle Neyman-Pearson detector (O. NPD): $|\boldsymbol{x}^\top \boldsymbol{s}_\ell| \gtrless_{\mathcal{H}_0}^{\mathcal{H}_1} \gamma$.
- A reduced one-dimension GLR test: $|\boldsymbol{x}^\top \boldsymbol{v}| \gtrless_{\mathcal{H}_0}^{\mathcal{H}_1} \gamma$, (*e.g.* $\boldsymbol{v}$: SVD of $\boldsymbol{S}$).

1) $\boldsymbol{s}_\ell$ similar shape to $\boldsymbol{v}$ ($\boldsymbol{s}_\ell^\top \boldsymbol{v} = 0.93$)



2) $\boldsymbol{s}_\ell$ dissimilar shape to $\boldsymbol{v}$ ($\boldsymbol{s}_\ell^\top \boldsymbol{v} = 0.50$)



R.D. test: advantageous w.r.t. *Max* test iff $\boldsymbol{s}_\ell$ is well correlated to the learned subspace.
$\Rightarrow$ possibility to devise robust (minimax) R.D. tests.

Model of reduced dimension and corresponding GLRT

Proposed R.D. model

$$\begin{cases} \mathcal{H}_0 & : \boldsymbol{x} = \boldsymbol{n}, \qquad \boldsymbol{n} \sim \mathcal{N}(\mathbf{0}, \mathbf{I}) \\ \mathcal{H}_1 & : \boldsymbol{x} = \boldsymbol{D}\beta + \boldsymbol{n} \quad ||\beta||_0 = 1 \end{cases}$$

$\boldsymbol{D} \in \mathbb{R}^{N \times K}$ with $||\boldsymbol{d}_j||_2^2 = 1$ is a low dimension ($K \ll L$) dictionary to be optimized.

Corresponding GLRT: *Max* test over *D*

$$T_{\boldsymbol{D}}(\boldsymbol{x}) = \max_{j=1,\dots,K} |\boldsymbol{d}_j^\top \boldsymbol{x}| \underset{\mathcal{H}_0}{\overset{\mathcal{H}_1}{\gtrless}} \xi.$$

Aim: optimize ***D*** to maximize the worst-case detection performance.

Minimax optimization criterion

Optimal minimax dictionary: exact criterion

$$\boldsymbol{D}^* = \arg\max_{\boldsymbol{D}} \min_{i=1,\dots,L} \mathbb{P}\left(\max_{j=1,\dots,K} |\boldsymbol{d}_j^\top \boldsymbol{x}| > \xi | \mathcal{H}_1, \boldsymbol{s}_i\right)$$

$$\text{subject to} \begin{cases} \mathbb{P}\left(\max_{j=1,\dots,K} |\boldsymbol{d}_j^\top \boldsymbol{x}| > \xi | \mathcal{H}_0\right) \leq \mathrm{P}_{\mathrm{FA}_0}, \\ \|\boldsymbol{d}_j\|_2 = 1,\ j = \{1,\dots,K\}. \end{cases}$$

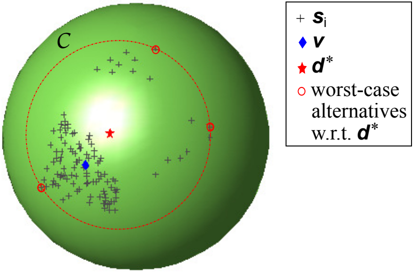
Solving $\boldsymbol{D}^* (K > 1)$ involves distributions of the max. of correlated variables $\boldsymbol{d}_j^\top \boldsymbol{x}$, $j = \{1,\dots,K\}$, which is an extremely intricate task.

Exact criterion for $K = 1$

$$\boldsymbol{d}^* = \arg\max_{\boldsymbol{d}:\|\boldsymbol{d}\|_2=1} \min_{i=1,\dots,L} |\boldsymbol{d}^\top \boldsymbol{s}_i|$$

Illustration of the minimax atom $\boldsymbol{d}^*$ $(K = 1)$

Proposition:
Assuming that $\boldsymbol{S} \in \mathbb{R}_+^N$, then $\boldsymbol{d}^* \in \mathbb{R}_+^N$ is the solution of a QP problem.



- On this example, $\boldsymbol{d}^*$ is held by three marginal alternatives (at the border of the smallest enclosed circle $\mathcal{C}$). Those "isolated" $\boldsymbol{s}_i$ induce the worst P_{Det}.
- $\boldsymbol{v}$ (SVD) represents well the most populated area of the alternatives.
- Representing $\boldsymbol{S}$ by a single atom may be insufficient w.r.t. the intrinsic diversity of $\boldsymbol{S}$.

Approximation criterion to optimize $\boldsymbol{D}$ for $K > 1$: deriving bounds

$$\mathrm{P}_{\mathrm{Det}}(\boldsymbol{s}_\ell, \boldsymbol{D}) = \mathbb{P}\left(\max_{j=1,\dots,K}(\boldsymbol{d}_j^\top \boldsymbol{x})^2 > \xi^2 | \mathcal{H}_1, \boldsymbol{s}_\ell, \boldsymbol{D}\right).$$

CDF of the max of continuous random variables $(X_1, \dots, X_N)$:
$F_{\max(X_1,\dots,X_N)}(t) \leq \min\left(F_1(t), \dots, F_N(t)\right).$

$$\mathrm{P}_{\mathrm{Det}}(\boldsymbol{s}_\ell, \boldsymbol{D}) \geq \max_{j=1,\dots,K} Q_{\frac{1}{2}}(|\boldsymbol{d}_j^\top \boldsymbol{s}_\ell|, \xi)$$

Approximation of the optimal dictionary

$$\boldsymbol{D}^* \approx \arg \max_{\boldsymbol{D}:\|\boldsymbol{d}_j\|_2=1} \rho^{(K)}(\boldsymbol{D})$$

where $\rho^{(K)}(\boldsymbol{D}) = \min_{i=1,\dots,L} \max_{j=1,\dots,K} |\boldsymbol{d}_j{}^\top \boldsymbol{s}_i|$ is the minimax correlation function.

Approximation criterion to optimize $\boldsymbol{D}$ for $K > 1$

$$\mathrm{P}_{\mathrm{FA}}(\boldsymbol{D}) = \mathbb{P}\left(\max_{j=1,\dots,K}(\boldsymbol{d}_j^\top \boldsymbol{x})^2 > \xi^2 | \mathcal{H}_0, \boldsymbol{D}\right).$$

CDF of multivariate normals (C. Khatri, 1968):

$$\mathbb{P}(|u_i| \leq c_i, i = 1, \dots, m) \geq \prod_{i=1}^{m} \mathbb{P}(|u_i| \leq c_i)$$

when $\boldsymbol{u} = (u_1, \dots, u_m)^\top$ is distributed as multivariate normal $\boldsymbol{u} \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma})$.

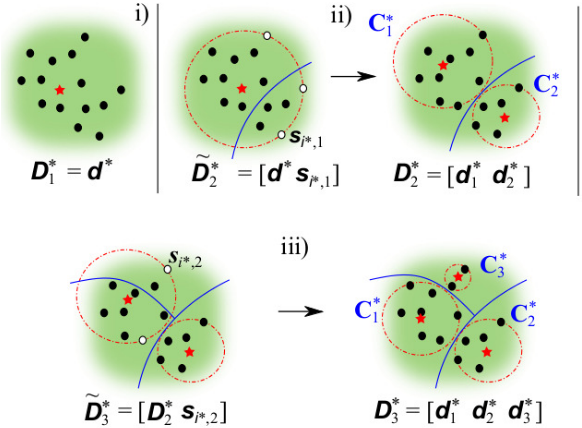
$$\mathrm{P}_{\mathrm{FA}}(\boldsymbol{D}) \leq 1 - \prod_{j=1}^{K} \mathbb{P}(|u_j| \leq \xi)$$

$$\mathrm{P}_{\mathrm{FA}}(\boldsymbol{D}) \leq 1 - \Phi_{\chi_1^2}^{K}(\xi^2).$$

If $\boldsymbol{D}$ is orthogonal, the upper bound is the exact $\mathrm{P}_{\mathrm{FA}}(\boldsymbol{D})$.

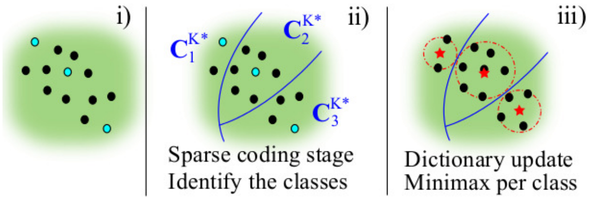
Ilustrations of the proposed learning methods to optimize *D*

- **Greedy minimax**: ($K = 3$)
  Heuristic optimization based on the approximation criterion. Dictionary update stage by 1-dimensional minimax optimization. Samples the distribution in a greedy manner to open new classes.



Ilustrations of the proposed learning methods to optimize *D*

- **K-minimax**: ($K = 3$)
  The dictionary update stage of the K-SVD algorithm [Aharon *et al.*, 2006] is replaced by the exact solution of the 1-dimensional minimax problem.



$\hookleftarrow$ ii)

Subspace learning of faces

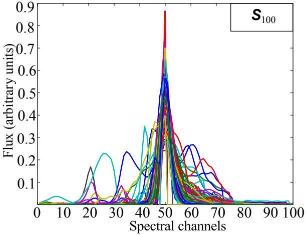
Library ***S***: 40 subjects in front position (possible alternatives under $\mathcal{H}_1$), taken from the ORL Database of Faces by AT&T Laboratories Cambridge.

K-SVD
($K = 3$)

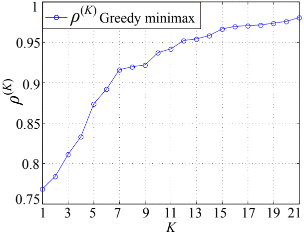
Greedy minimax
($K = 3$)

Minimax approach captures marginal features, (K)-SVD tends to represent “average face(s)”.

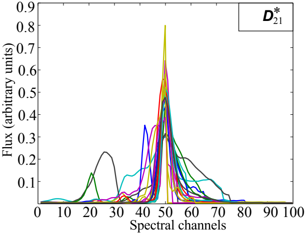
Subspace learning of 100 spectral profiles



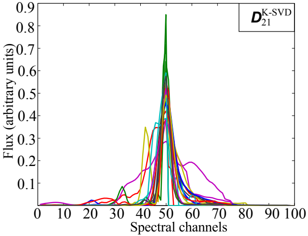
i. Spectral profiles, $\boldsymbol{S} \in \mathbb{R}^{100 \times 100}$



ii. Minimax correlation function, $\rho^{(K)}$

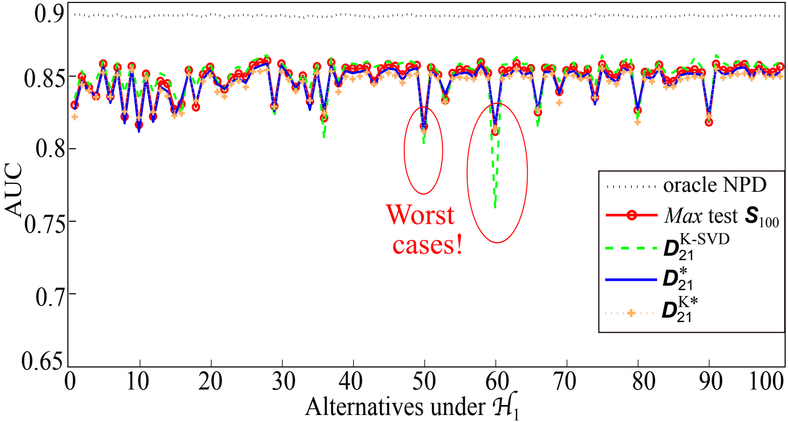


iv. $\boldsymbol{D}^*_{21}$ (minimax)



v. $\boldsymbol{D}^{\text{K-SVD}}_{21}$

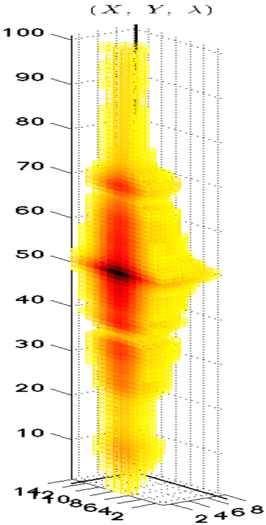
AUC (Area under curve) of the ROCs

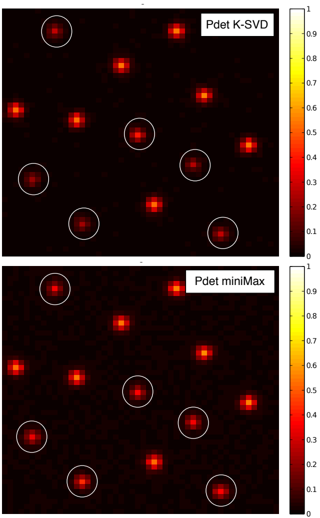


Performances ranking: Minimax vs Average

Dictionary	Ranking Criterion	
	min AUC	Average AUC
O. NPD	Ref : 0.886	Ref : 0.887
$\boldsymbol{S}_{100}$	1st : 0.813	3rd : 0.847
$\boldsymbol{d}^*$	2nd : 0.794	6th : 0.836
$\boldsymbol{v}$ (SVD)	4th : 0.699	1st : 0.863
$\boldsymbol{D}_{21}^{\text{K-SVD}}$	3rd : 0.764	2nd : 0.849
$\boldsymbol{D}_{21}^*$	1st : 0.812	4th : 0.845
$\boldsymbol{D}_{21}^{\text{K}*}$	1st : 0.813	5th : 0.843

MUSE simulation. 3D atoms, SNR = -11 dB, $P_{FA} = 0.01$





Conclusions

- Propose a detection test of reduced dimension where dictionary $\boldsymbol{D}$ is learned with the objective of maximizing the worst $\mathrm{P_{Det}}$.
- Case $K = 1$, exact solution to the minimax problem, but might be insufficient to represent well all the intrinsic diversity of $\boldsymbol{S}$.
- Case $K > 1$:
  - We derived proxies that are used to elaborate two algorithms for minimax detection.
  - better sampling of $\boldsymbol{S}$, thus improves the minimax performance over the case $K = 1$.

Perspectives

- Minimax modification of learning algorithm involving sparsity can be easily achieved by the use of use of $\boldsymbol{d}^*$ in the dictionary update stage.
- Find the optimal value of K, which depends on the intrinsic diversity of $\boldsymbol{S}$.

Thank you

Loss of performances of *Max* test as L increases

Assuming that $\boldsymbol{S} \in \mathbb{R}^{N\times L}$ is orthonormal ($\boldsymbol{s}_i^\top \boldsymbol{s}_j = \delta_{i,j}$) and that the active alternative under $\mathcal{H}_1$ is $\boldsymbol{s}_\ell$ (we assume that $\alpha = 1$), one can prove that:

$$\mathrm{P_{Det}}(\boldsymbol{s}_\ell) = 1 - \Phi_{\chi^2_{1,1}}\left(\Phi^{-1}_{\chi^2_1}\left((1-\mathrm{P_{FA}})^{\frac{1}{L}}\right)\right)\,(1-\mathrm{P_{FA}})^{\frac{L-1}{L}}.$$

(Φ_ν is the cumulative distribution function of ν).

- As L grows, the *Max* test correlates increasingly many alternatives to the data, with only marginal improvement of $\mathrm{P_{Det}}$ but substantially increased of $\mathrm{P_{FA}}$.
- Asymptotically, the GLR has no power in this setting.

$$\lim_{L\to+\infty} \Phi_{\chi^2_{1,1}}\left(\Phi^{-1}_{\chi^2_1}\left((1-\mathrm{P_{FA}})^{\frac{1}{L}}\right)\right) = 1,$$

$$\lim_{L\to+\infty} \mathrm{P_{Det}}(\boldsymbol{s}_\ell) = \mathrm{P_{FA}},\ \forall \ell : 1,\dots,L.$$