# A geometrical blind separation method for unconstrained-sum locally dominant sources

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# Summary

1. Blind Source Separation
2. Geometrical framework
3. MASS
4. Hyperspectral unmixing in astro-physics
5. Experimental results

# Blind Source Separation

- A generic signal processing problem

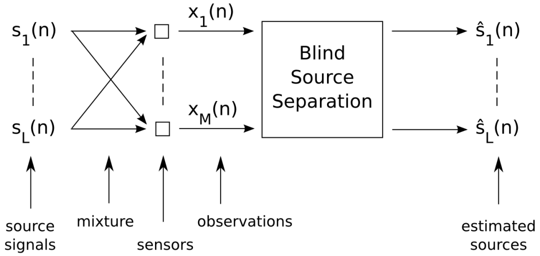
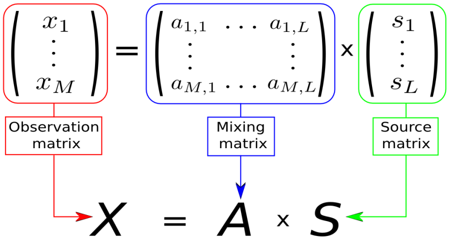



Figure : Blind source separation problem (L sources and M observations ($M \geq L$)).

- Different mixing operators: linear instantaneous, anechoic, convolutive, non-linear
- Different signals:
  - 1D (audio, communication, spectroscopy...)
  - 2D (images)

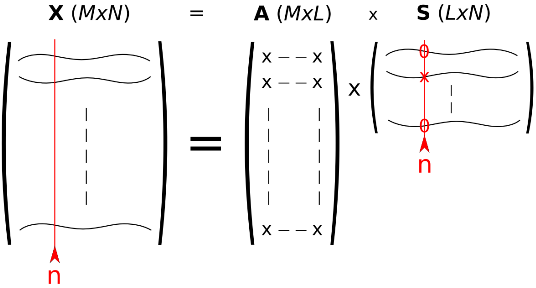
# Linear Blind Source Separation

- Instantaneous linear mixing: $x(m,n) = \sum_{\ell=1}^{L} a(m,\ell) \times s(\ell,n)$




- 3 classes of methods to solve the linear problem:
  - Independent Component Analysis (ICA)
  - Non-Negative Matrix Factorization (NMF)
  - Sparse Component Analysis (SCA)
- Subclass of methods: NMF + SCA $\rightarrow$ geometric methods

- Single-source observations:




- Sparsity assumption: for each source, there exist at least one sample index *n* of observations for which these source is non-zero.

- Sum-to-one constraint:
  - In many geometric methods (hyperspectral image unmixing in Earth observation):

$$\sum_{\ell=1}^{L} a(m,\ell) = 1 \qquad \forall m \in \{1,\ldots,M\} \tag{1}$$

# Linear Blind Source Separation

- Sum-to-one constraint:
  - for our applications:

$$\sum_{\ell=1}^{L} a(m,\ell) = 1 \qquad \forall m \in \{1,\dots,M\} \tag{1}$$

- Assumptions:
  - $X$,$A$ and $S$ are non-negative
  - $A$ is a full column rank matrix
  - The number of sources $L$ is known
  - Sparsity assumption of sources (presence of single source observations)
  - The sum of mixing coefficients is unconstrained

# Geometrical framework

- Each observed vector $x_m$ is represented as an element of a $\mathbb{R}^N$ vector space.

$$x_m = As_m \tag{2}$$

- $A$ and $S$ being non-negative and the column $a_\ell$ being linearly independent:
- The set:

$$\mathcal{C}_A = \{x_m \mid x_m = As_m,\ s_m \in \mathbb{R}_+^L\} \tag{3}$$

  is a simplicial cone (the convex hull spanned by the non negative linear combination of the columns of A).
- Each column vector $a_\ell$ of $A$ spans an edge $\mathcal{E}_\ell$ of the simplicial cone $\mathcal{C}_A$:

$$\mathcal{E}_\ell = \{c \mid c = \alpha a_\ell,\ \alpha \in \mathbb{R}_+\} \tag{4}$$

# Geometrical framework

- 2-dimension example:

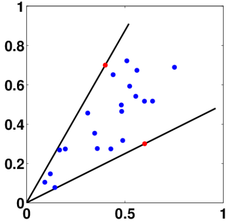



Figure : Scatter plot of mixed data and edges of the simplicial cone.

- If the sparsity assumption of sources is valid:

$$\mathcal{C}_A = \mathcal{C}_X \quad (5)$$

- Identifying the column of $A$ $\rightarrow$ finding the columns of $X$ which are furthest apart in the angular sense.

# Maximum Angle Source Separation (MASS)

- Estimation of mixing matrix $A$: L-1 steps
  - Columns of $X$ are normalize to unit length.
  - Illustration of the algorithm in 3-dimension with a mixture of 3 sources:

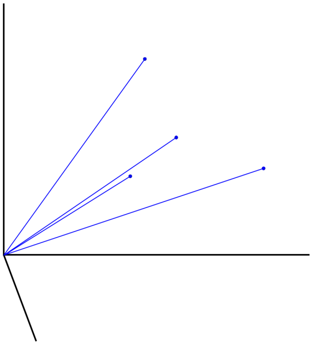


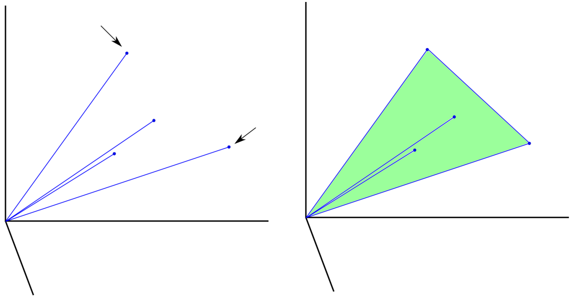
Figure : Scatter plot of the observed data.

# Maximum Angle Source Separation (MASS)

1. Identify the first 2 columns of $\hat{A}$ by selecting the two columns of $X$ that have the largest angle:

$$(m_1, m_2) = \underset{i,j}{\operatorname{argmax}} \cos^{-1}(x_i^T x_j) \qquad \forall i, j \in \{1, \ldots, M\}. \tag{6}$$

$$(m_1, m_2) = \underset{i,j}{\operatorname{argmin}}\, x_i^T x_j \qquad \forall i, j \in \{1, \ldots, M\}. \tag{7}$$



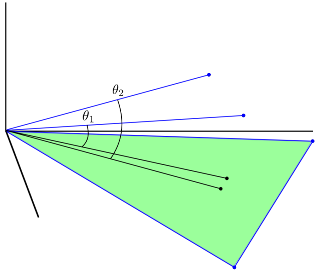
$$\tilde{A} = [x_{m_1}, x_{m_2}] \tag{8}$$

# Maximum Angle Source Separation (MASS)

2. Identify the column which has the largest angle with $x_{m_1}$ and $x_{m_2}$:
   - maximum angle between the column and its orthogonal projection on the simplicial cone spanned by the columns of $\tilde{A}$:

$$\Pi_{\tilde{A}}(X) = \tilde{A}(\tilde{A}^T\tilde{A})^{-1}\tilde{A}^T X. \tag{9}$$

$$m_3 = \underset{i}{\operatorname{argmin}}\, x_i^T \pi_i \qquad \forall i \in \{1, \ldots, M\} \tag{10}$$

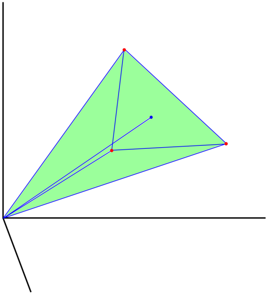



$$\tilde{A} = [x_{m_1}, x_{m_2}, x_{m_3}] \tag{11}$$

# Maximum Angle Source Separation (MASS)

3. This projection and identification procedure is then repeated to identify the $L$ columns of the mixing matrix.

$$\hat{A} = [x_{m_1}, \ldots, x_{m_L}] \tag{12}$$



- Source matrix reconstruction:
  $X$, $A$ and $S$ are non-negative $\rightarrow$ Non-Negative Least Square algorithm (NNLS):

$$J(\hat{s}_m) = \frac{1}{2}\|x_m - \hat{A}\hat{s}_m\|_2^2 \qquad s.t.\ \hat{s}_m \geq 0, \quad \forall m \in \{1, \ldots, M\} \tag{13}$$

# Hyperspectral unmixing in astrophysics

- Application field of MASS (non-negativity, correlated sources without sum-to-one constraint )is very common in astrophysics.
- Area observed at high spectral resolution → each pixel corresponds to an emission spectrum of a portion of the area.

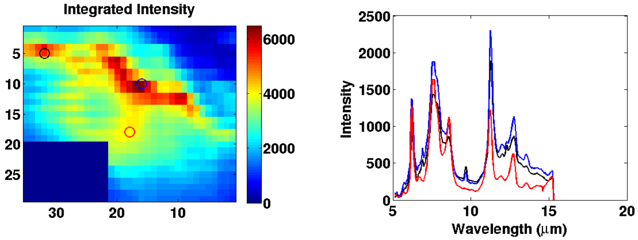
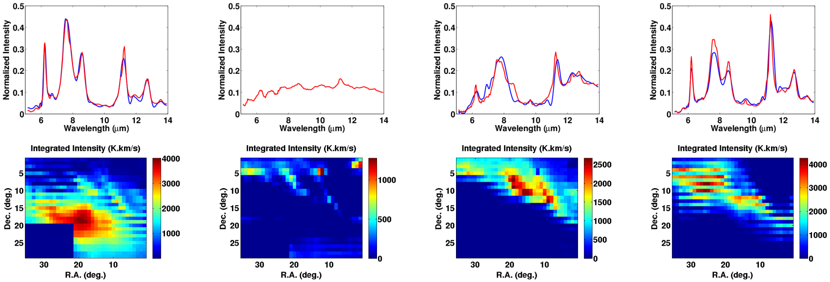



Figure : NGC7023

Problem: The observed spectra are generally constituted by a mixture of elementary spectra (components of the gas cloud containing different chemical species).

# Experimental results




(a) Source 1 (b) Source 2 (c) Source 3 (d) Source 4

Figure : Extracted spectra in NGC7023-NW: NMF → blue spectra, MASS → red spectra.

- Similar results for the extracted spectra
- Very fast algorithm compared to the NMF
- Uniqueness of the solution
- MASS is able to identify a weak signal present in only some observation