Groupe hyperspectral

Hyperspectral Unmixing Geometrical, Statistical, and Sparse Regression-Based Approaches

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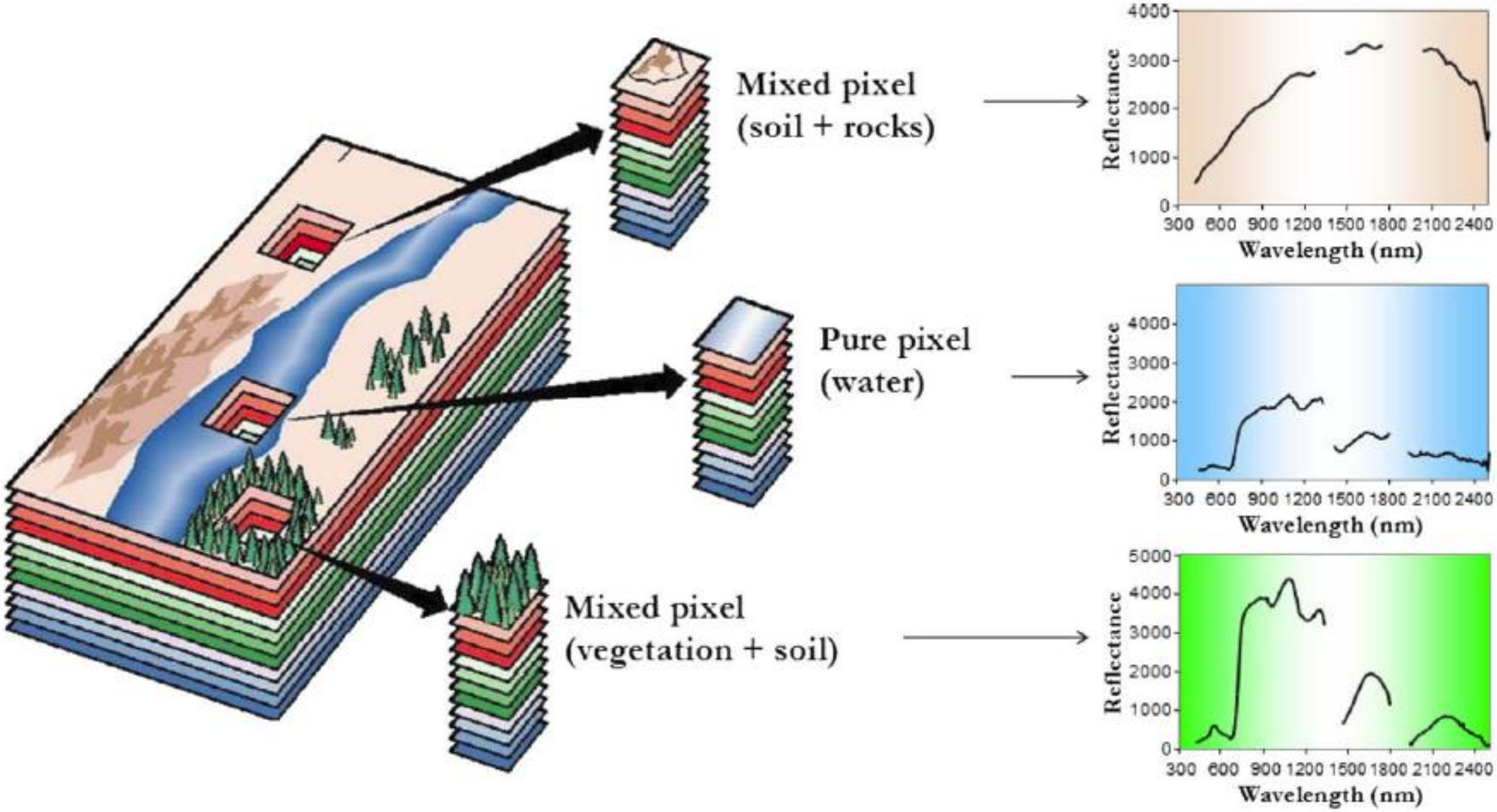
Hyperspectral imaging (and mixing)



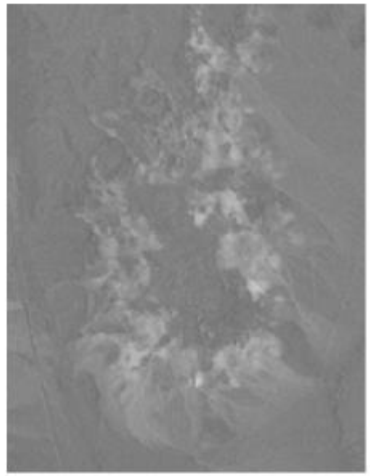
Hyperspectral unmixing

AVIRIS of Cuprite,
Nevada, USA

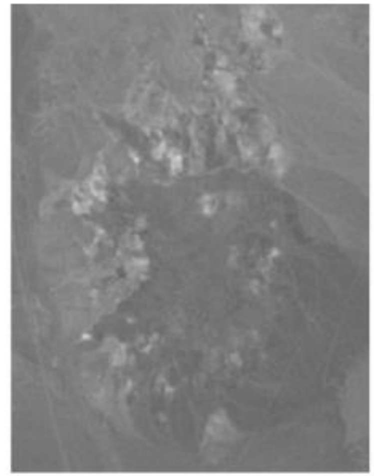
R – ch. 183 (2.10 μm)

G – ch. 193 (2.20 μm)

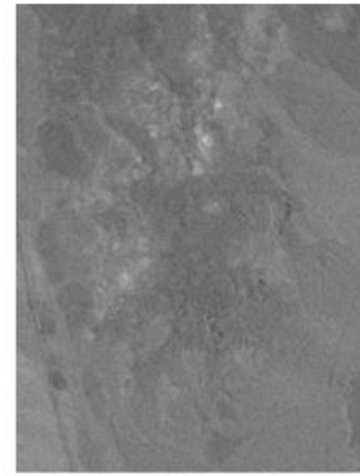
B – ch. 207 (2.34 μm)



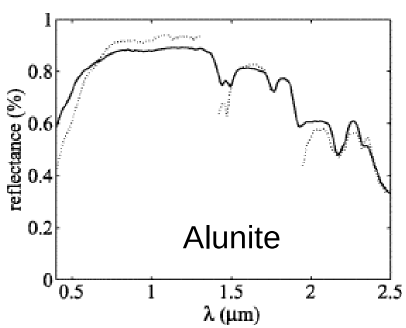
(a)



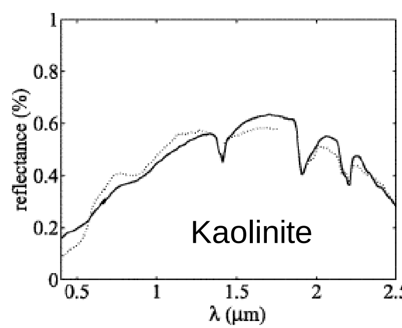
(b)



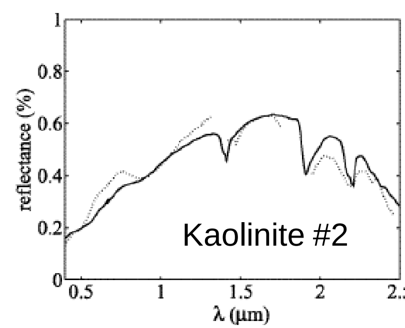
(c)



(a)



(b)



(c)

VCA [Nascimento, B-D, 2005]

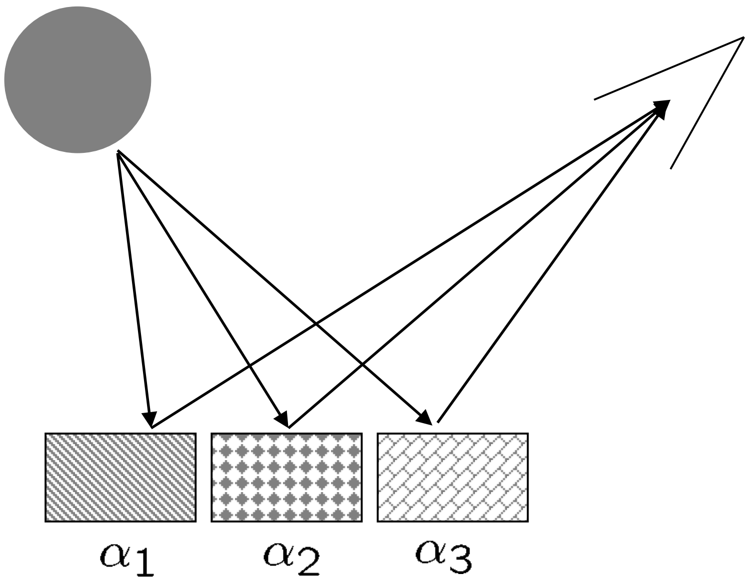
Outline

- Mixing models
  - Linear
  - Nonlinear
- Signal subspace identification
- Unmixing
  - Geometrical-based
  - Statistical-based
  - Sparse regression-based

J. M. Bioucas-Dias, A. Plaza, N. Dobigeon, M. Parente, Q. Du, P. Gader, and J. Chanussot, "Hyperspectral unmixing overview: geometrical, statistical, and sparse regression-based approaches", IEEE Journal of Selected Topics in Applied Earth Observations and Remote Sensing, vol. 5, no. 2, pp. 354-379, 2012.

W.-K. Ma, J. Bioucas-Dias, T.-H. Chan, N. Gillis, P. Gader, A. Plaza, A. Ambikapathi and C.-Y. Chi, "A signal processing perspective on hyperspectral unmixing", IEEE Signal Processing Magazine, Jan., 2014.

Linear mixing model (LMM)



Incident radiation interacts only with one component (checkerboard type scenes)

$$\mathbf{r} = \sum_{i=1}^{p} \alpha_i \mathbf{m}_i \qquad \mathbf{m}_i = \begin{bmatrix} \rho_{i1} \\ \rho_{i2} \\ \vdots \\ \rho_{iL} \end{bmatrix}$$

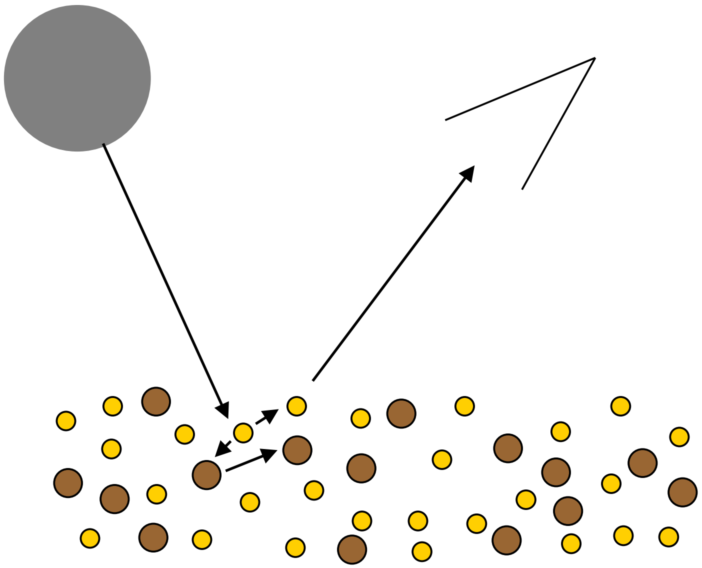
$$\mathbf{r} = \mathbf{M}\boldsymbol{\alpha}$$

$$\mathbf{M} \equiv [\mathbf{m}_1, \mathbf{m}_2, \mathbf{m}_3] \qquad \boldsymbol{\alpha} = \begin{bmatrix} \alpha_1 \\ \alpha_2 \\ \alpha_2 \end{bmatrix}$$

Hyperspectral linear unmixing → Estimate $\mathbf{M}, \ \boldsymbol{\alpha}$

Nonlinear mixing model

Intimate mixture (particulate media)

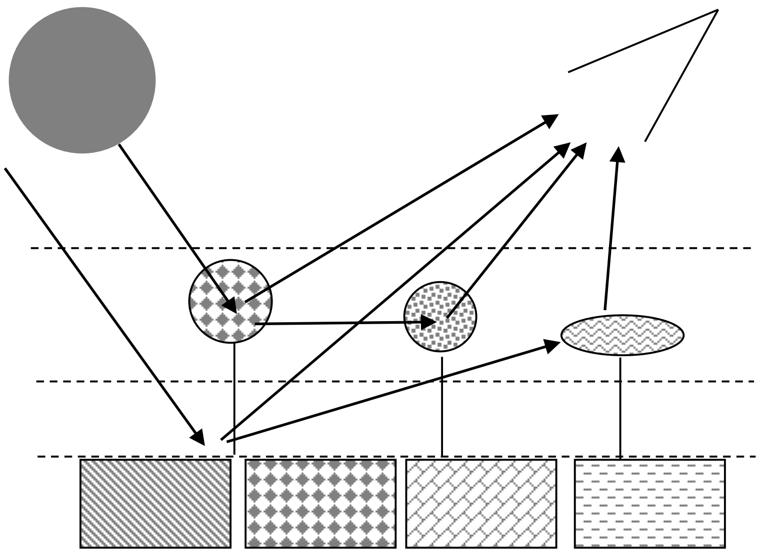


Radiative transfer theory

$$\mathbf{r} = f(\alpha, \theta)$$

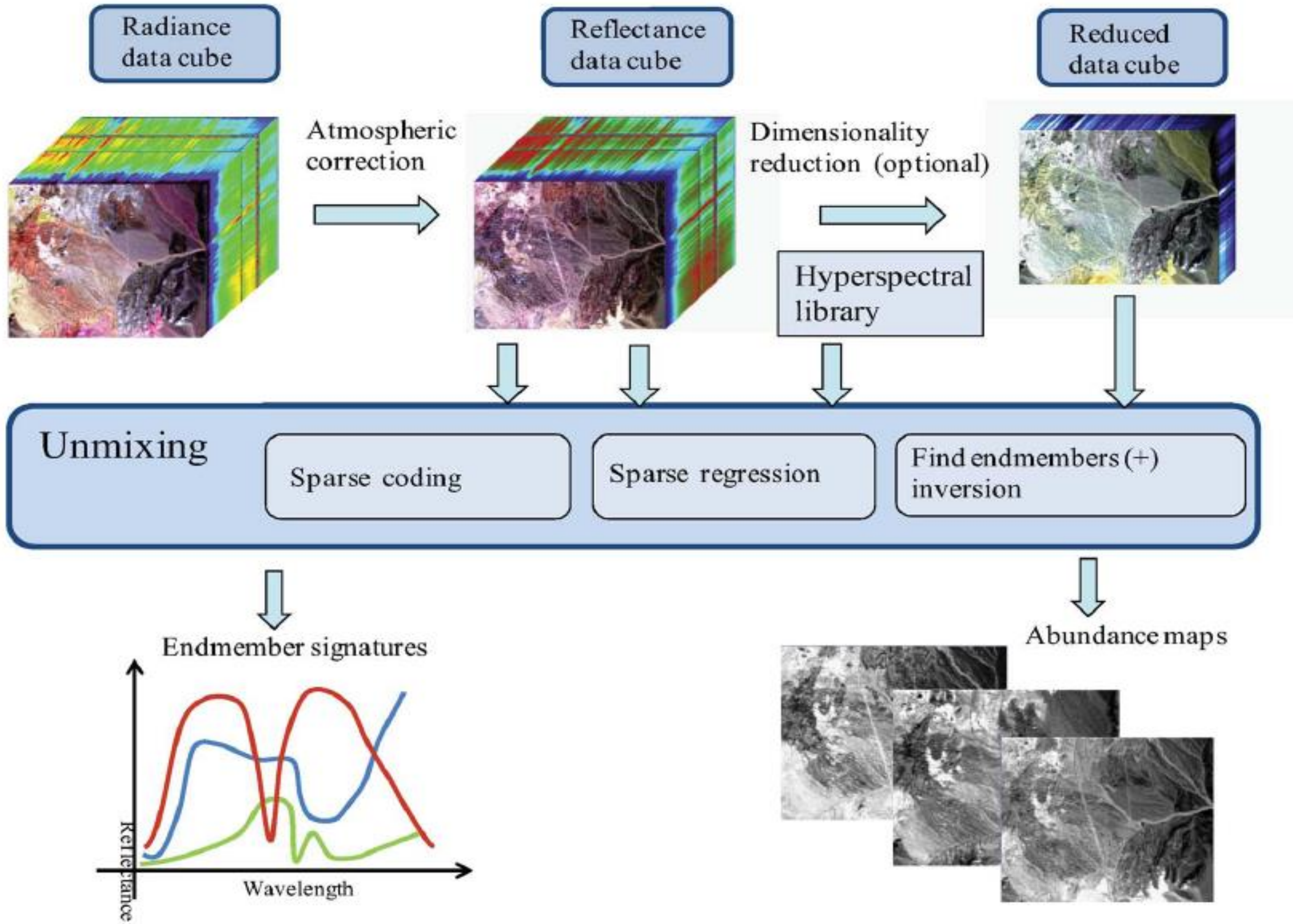
material fractions media parameters

Two-layers: canopies+ground



$$\mathbf{r} = \underbrace{\sum_{i=1}^{p} \alpha_i \mathbf{m}_i}_{\text{single scattering}} + \underbrace{\sum_{\substack{i,j=1 \\ i \neq j}}^{p} \alpha_{i,j} \mathbf{m}_i \odot \mathbf{m}_j}_{\text{double scattering}}$$

Schematic view of the unmixing process



Spectral linear unmixing (SLU)

Given N spectral vectors of dimension L:

$$\mathbf{Y} = \left\{ \mathbf{y}_i \in \mathbb{R}^L,\ i = 1, \ldots, N \right\}$$

ANC: abundance nonnegative constraint

Subject to the LMM: $\mathbf{y} = \mathbf{M}\boldsymbol{\alpha} + \mathbf{n},\quad \boldsymbol{\alpha} \geq 0,\ \mathbf{1}^T\boldsymbol{\alpha} = 1$

ASC: abundance sum-to-one constraint

Determine:

- The mixing matrix $\mathbf{M}$ (*endmember spectra*)
- The *fractional abundance* vectors $\boldsymbol{\alpha}$

$\Rightarrow$ SLU is a blind source separation problem (BSS)

Subspace identification

$$\mathbf{y} = \mathbf{M}\boldsymbol{\alpha} + \mathbf{n} \qquad \mathrm{dim}(\mathbf{M}) = [L \times p] \qquad L \gg p$$

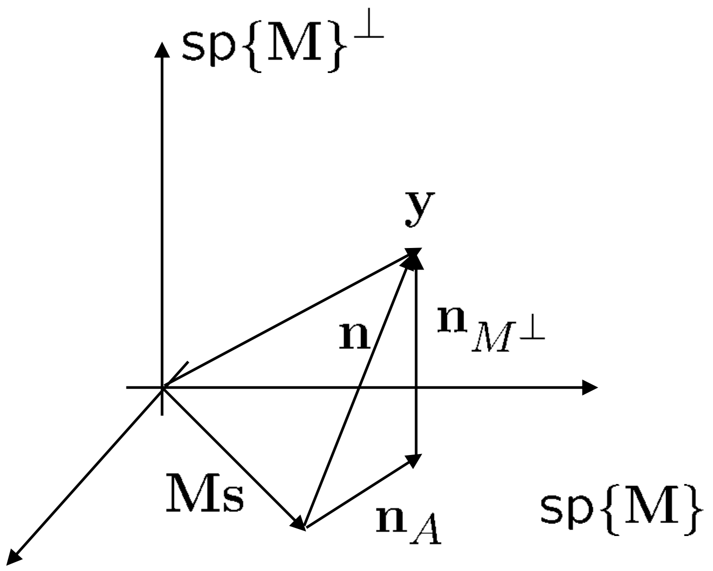
Problem: Identify $\mathrm{span}\{\mathbf{M}\}$

the subspace generated by the columns of $\mathbf{M}$

Reasoning underlying DR

1. Lightens the computational complexity

2. Attenuates the noise power by a factor of p/L



Subspace identification algorithms

Exact ML solution [Scharf, 91] (known *p*, i.i.d. Gaussian noise)

PCA - Principal component analysis (unknown *p*, i.i.d. noise)

NAPC - Noise adjusted principal components [Lee *et al*., 90]

MNF - Maximum noise fraction [Green *et al.*, 88]

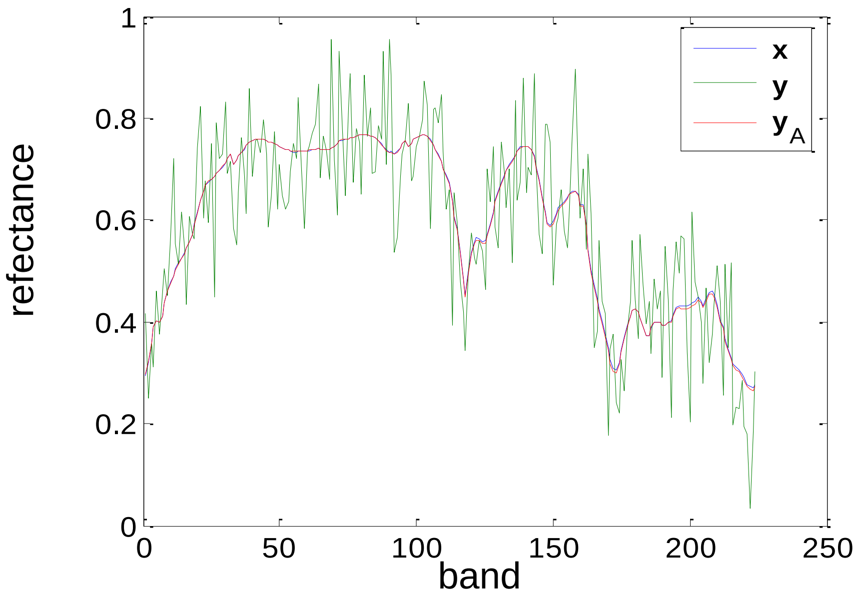
HFC - Harsanyi-Farrand-Chang [Harsanyi *et al*., 93]

NWHFC - [Chang, Du, 94]

HySime - Hyperspectral signal identification by minimum error [B-D, Nascimento, 08]

GENE - geometry-based estimation of number of endmembers [ArulMurugan, 13]

RMT - [Kritchman, Nadler, 2009], [Cawse et al., 11], [Halimi et al., 16]



Example (HySime)

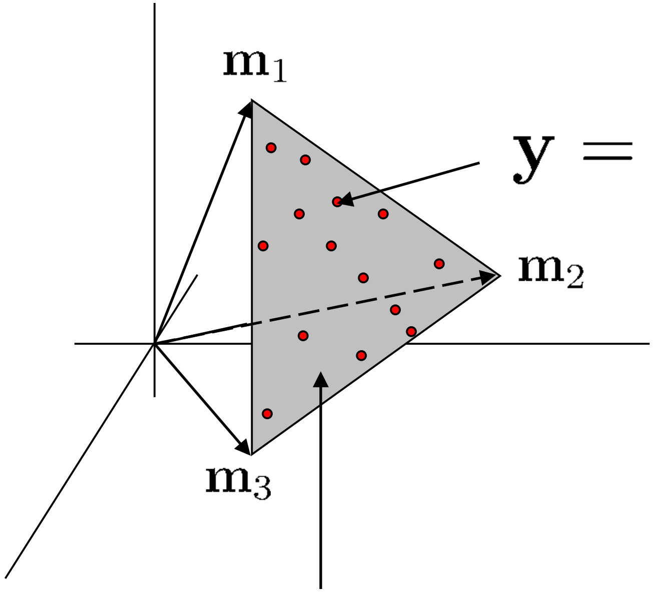
$$L = 224 \quad p = 3$$

$$\mathbf{n} \sim \mathcal{N}(\mathbf{0}, \sigma^2\mathbf{I})$$

$$\sigma = 0.1$$

Geometrical view of SLU

$$\mathbf{M} = [\mathbf{m}_1, \ldots, \mathbf{m}_p]$$



$$\mathbf{y} = \mathbf{M}\boldsymbol{\alpha}$$

$$\sum_{j=1}^{p} \alpha_j = 1, \ \alpha_j \geq 0$$

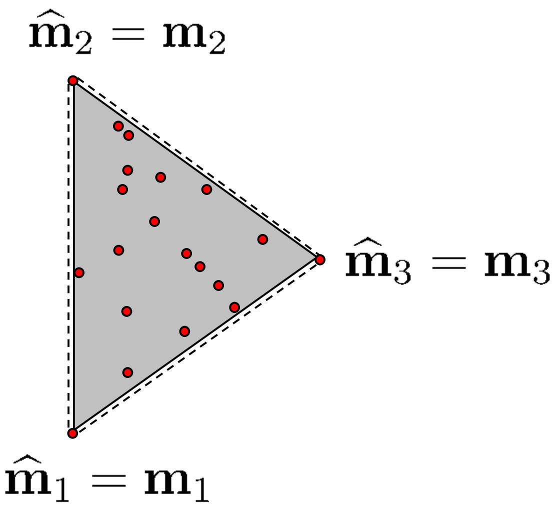
probability simplex (S_I)

$$S_M = \{\mathbf{x} \in \mathbb{R}^p \ : \ \mathbf{x} = \mathbf{M}\boldsymbol{\alpha}, \ \boldsymbol{\alpha} \in S_I\}$$ ➡ (p-1) - simplex

Inferring **M** $\Leftrightarrow$ inferring the vertices of the simplex S_M

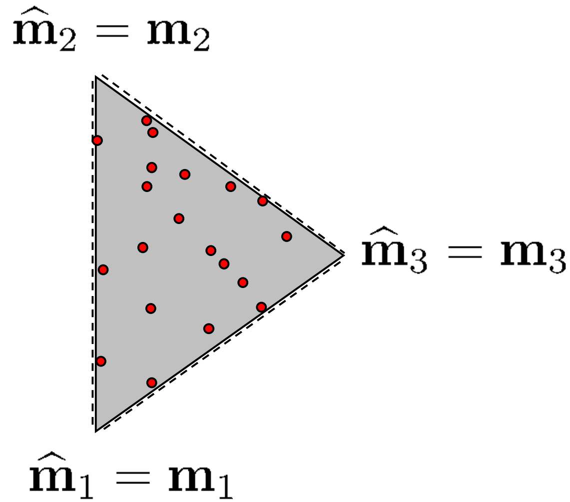
Classes of SLU problems

Pure pixels



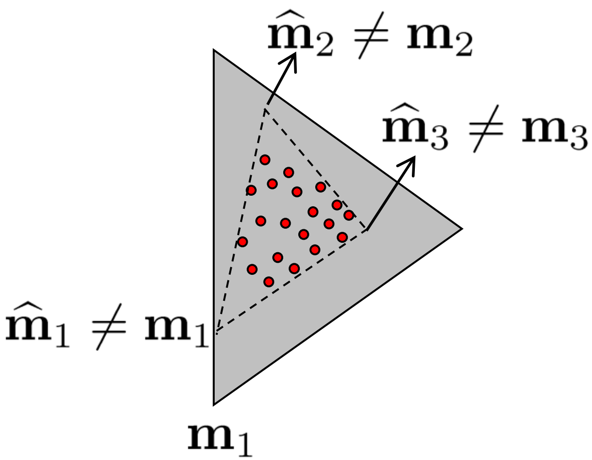
Well posed

Pixels in the facets



Well posed

Highly mixed



Ill-posed

Algorithms		
Vertex pursuit	Facet estimation	Statistical inference Sparse regression

Unmixing frameworks

- Geometrical (blind)

  Exploits parallelisms between the linear mixing model and properties of convex sets

  Application scenarios: pure pixels, pixels in facets

- Statistical (blind, semi-blind)

  Approaches linear unmixing as a statistical inference problem

  Application scenarios: all

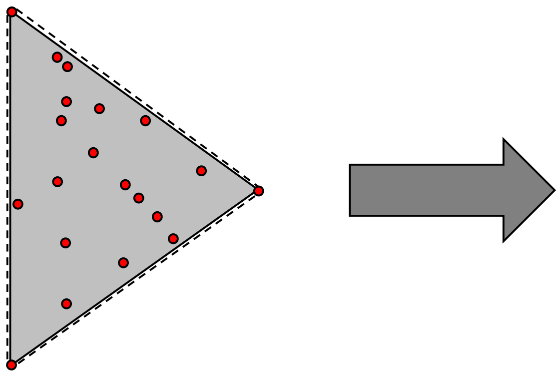
- Sparse regression (semi-blind)

  Approaches linear unmixing as a sparse regression problem

  Application scenarios: all

Simplex vertex pursuit

Hard assumption
The data set contains at least one pure pixel of each material



- Search endmembers in the data set
- Computationally light

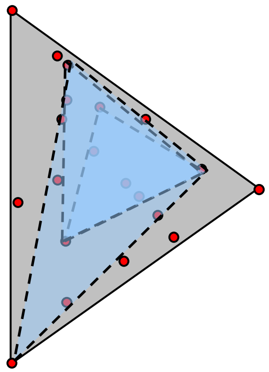
PPI - [Boardman, 93]; **N-FINDR** - [Winter, 99]; **IEA** - [Neville *et al*., 99];
AMEE – [Plaza *et al*, 02]; **SMACC** – [Gruninger *et al.*, 04]
VCA - [Nascimento, B-D, 03, 05]; **SGA** - [Chang *et al*., 06]
AVMAX, **SVMAX** - [Chan, et al., 11]; **RNMF**- [Gillis & Vavasis, 12,14];
SD-SOMP, **SD-ReOMP** - [Fu et al.13, 15];

Simplex vertex pursuit

N-FINDR

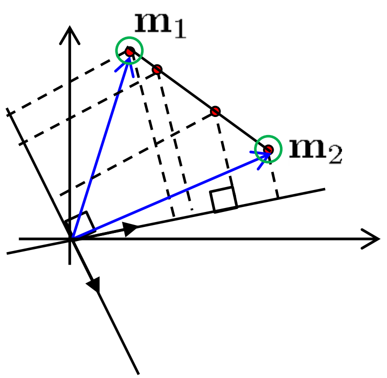
$$\text{iteratively increase } |\det(\mathbf{m}_1, \ldots, \mathbf{m}_p)|$$

$$\mathbf{m}_1, \ldots, \mathbf{m}_p \in \{\mathbf{y}_1, \ldots, \mathbf{y}_N\}$$



AVMAX

$$\begin{aligned} \max_{\mathbf{m}_1,\ldots,\mathbf{m}_p} \quad & |\det(\mathbf{m}_1, \ldots, \mathbf{m}_p)| \\ \text{st:} \quad & \mathbf{m}_1, \ldots, \mathbf{m}_p \in \text{conv}\{\mathbf{y}_1, \ldots, \mathbf{y}_N\} \end{aligned}$$



VCA

for $k = 1 : p$

$$\mathbf{m} := \arg\max_{\mathbf{y}_i} \|(P_{\mathbf{M}}^{\perp}\xi)^T \mathbf{y}_i\|$$

$$\mathbf{M} := [\mathbf{M},\ \mathbf{m}]$$

SVMAX

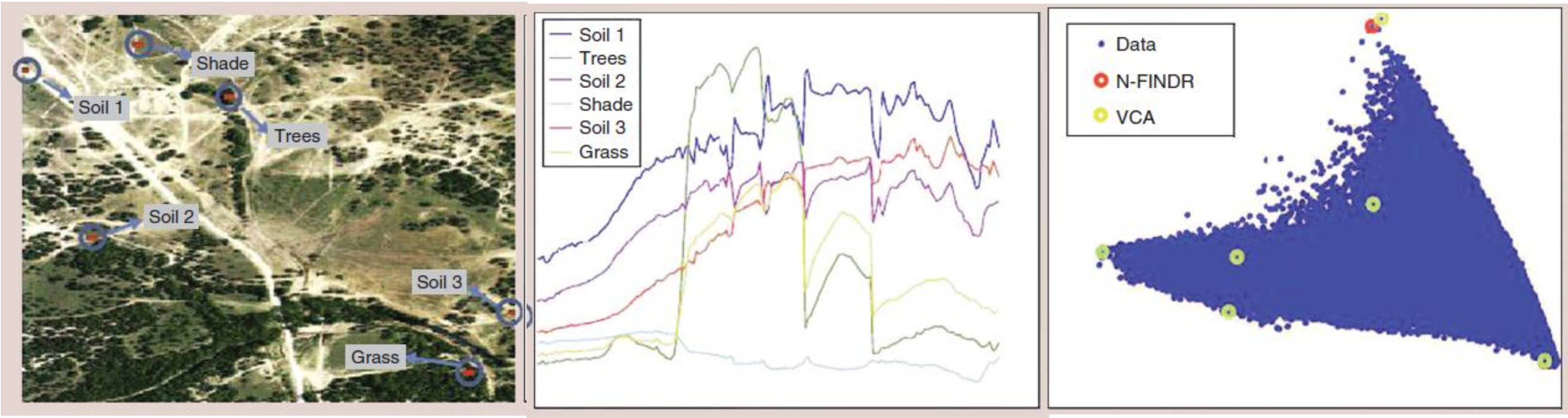
for $k = 1 : p$

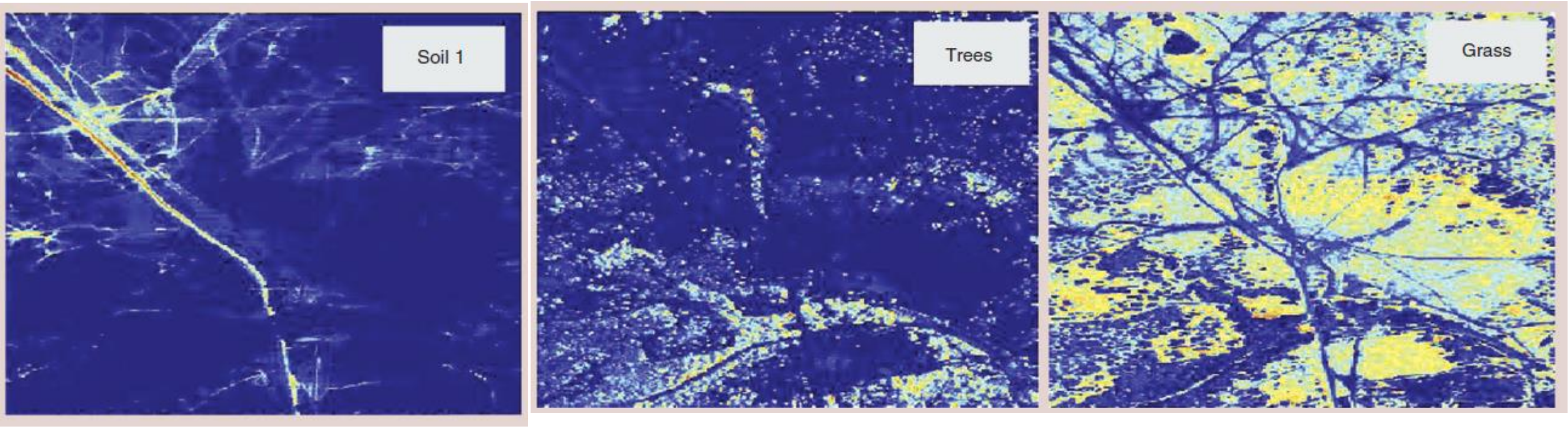
$$\mathbf{m} := \arg\max_{\mathbf{y}_i} \|(P_{\mathbf{M}}^{\perp}\mathbf{y}_i\|$$

$$\mathbf{M} := [\mathbf{M},\ \mathbf{m}]$$

Unmixing example

HYDICE sensor $\mathbf{M} \in \mathbb{R}^{210\times 6}$ $N = 500 \times 307$ resolution $= 0.75\text{m}$





Simplex facet estimation

- Minimum-volume constrained nonnegative matrix factorization (MVC-NMF) (inspired by NMF [Lee, Seung, 01])

$$(\widehat{\mathbf{M}}, \widehat{\mathbf{S}}) = \arg \min_{\mathbf{M}\in\mathbb{R}^{L\times p}, \mathbf{S}\in\mathbb{R}^{p\times N}} \frac{1}{2}\|\mathbf{Y} - \mathbf{MS}\|_F^2 + \tau V(\mathbf{M}) + \lambda\phi(\mathbf{S})$$

$$\text{s.t.:} \quad \mathbf{M} \succeq 0, \quad \mathbf{S} \succeq 0, \quad \mathbf{1}^T\mathbf{S} = \mathbf{1}_N^T,$$

volume regularizer

DPFT - [Craig, 90]; **CCA** - [Perczel et al., 89] (seminal works on MVC)

ICE - [Breman *et al*., 2004] ($V(\mathbf{M}) \equiv \text{quadratic}, \ \phi = 0$);

MVC-NMF - [Miao,Qi, 07] ($V(\mathbf{M}) = |\det(\mathbf{M}\mathbf{M}^T)|, \ \phi = 0$);

SPICE - [Zare, Gader, 2007] ($V(\mathbf{M}) \equiv \text{quadratic}, \ \phi \equiv \text{weighted } \ell_1$)

L1/2 - NMF - [Qian, Jia, Zhou, Robles-Kelly, 11] ($V = 0$, $\phi(\mathbf{S}) = \sum_{ij} |\alpha_{ij}|^{1/2}$)

CoNMF - [Li, B-D, Plaza, 12] ($V(\mathbf{M}) \equiv \text{quadratic}, \ \phi(\mathbf{S}) = \|\mathbf{S}\|_{2,1}$)

Minimum volume simplex algorithms

MVSA – Minimum volume simplex analysis [Li, B-D, 08]

Optimization variable $\mathbf{Q} \equiv \mathbf{M}^{-1} \quad \Rightarrow \quad \mathbf{QY} = \mathbf{S}$

$$\widehat{\mathbf{Q}} = \arg\max_{\mathbf{Q}} \log(|\det(\mathbf{Q})|)$$

$$\text{s.t.:} \quad \underbrace{\mathbf{1}_p^T \mathbf{Q} = \mathbf{q}_m}_{\text{ASC}}, \quad \underbrace{\mathbf{QY} \geq 0}_{\text{ANC}}$$

MVSA solves a sequence of quadratic programs

MVES [Chan, Chi, Huang, Ma, 2009]

- Solves a sequence of linear programs by exploiting the cofactor expansion of det (**Q**)
- The existence of pure pixels is a sufficient condition for exact identification of the true endmembers

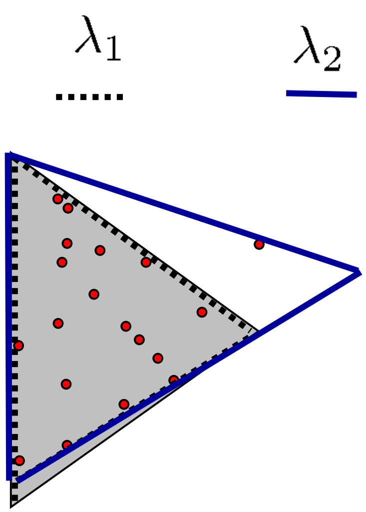
Robust minimum volume simplex algorithms: outliers

SISAL – Simplex identification via split augmented Lagrangian [B-D,09]

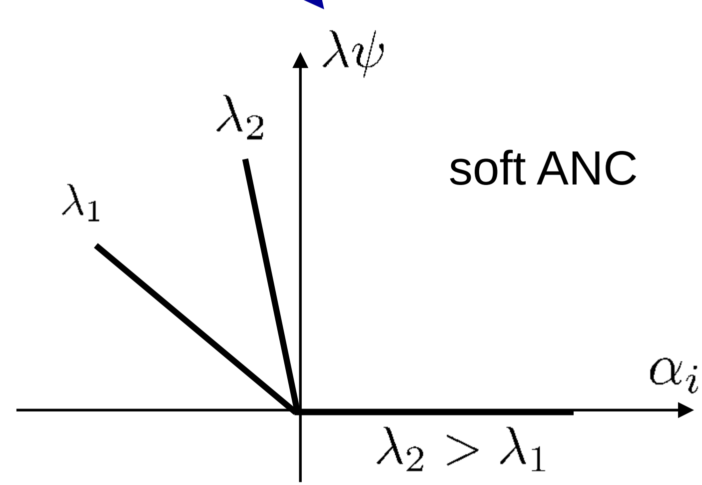
$$\widehat{\mathbf{Q}} = \arg\max_{\mathbf{Q}} \log(|\det(\mathbf{Q})|) - \lambda\phi(\mathbf{QY})$$

$$\text{s.t.:} \quad \mathbf{1}_p^T\mathbf{Q} = \mathbf{q}_m$$

ASC



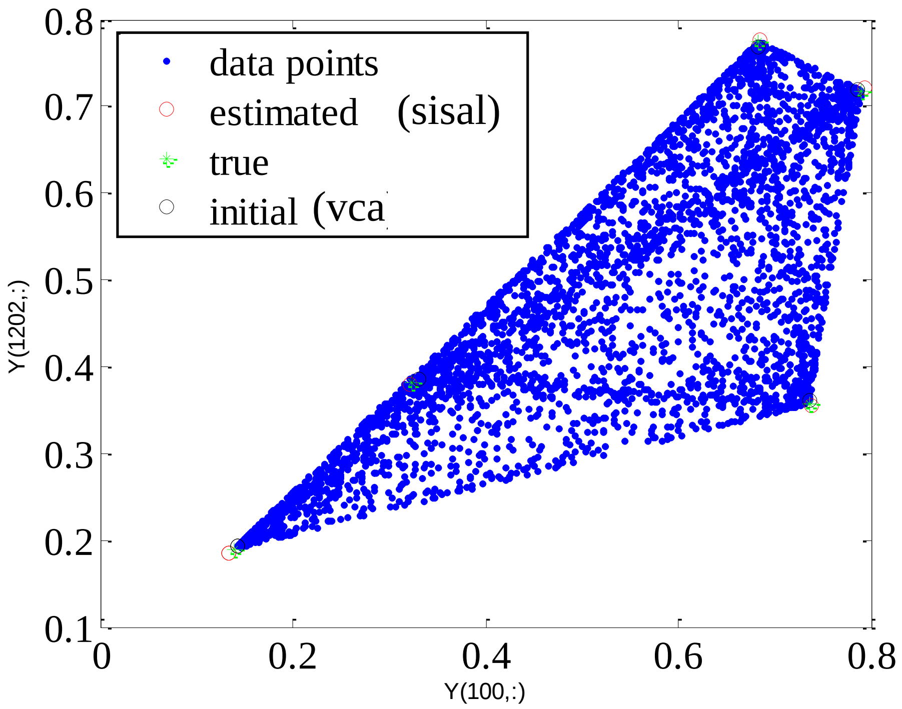
$\lambda_2 > \lambda_1$

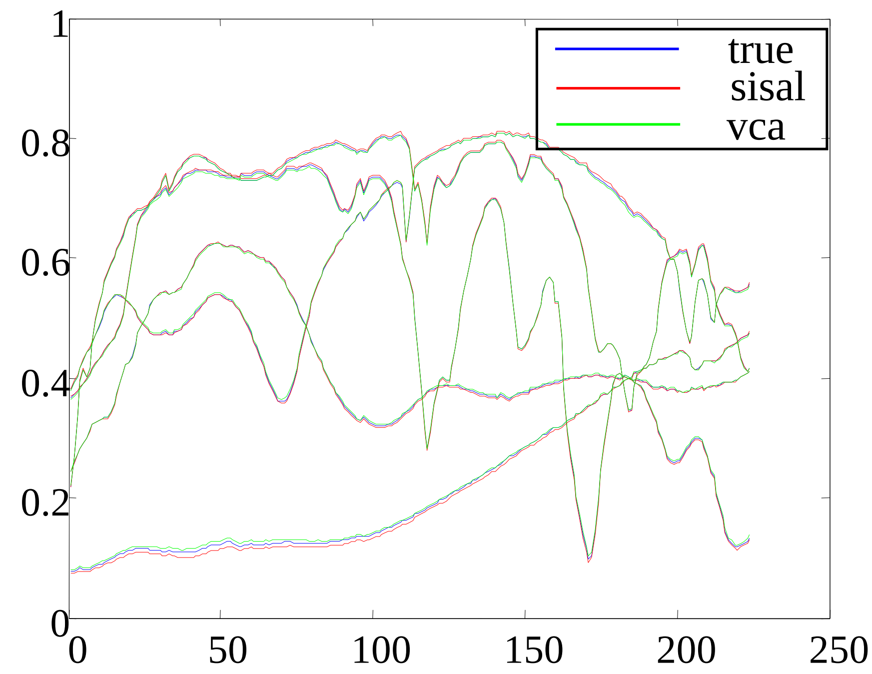


- **SISAL** solves a sequence of convex subproblems using ADMM
- $(\lambda = \infty) \Rightarrow \mathbf{MVES} \equiv (\mathbf{MVES},\ \mathbf{SISAL})$

Example: data set contains pure pixels

$$N = 5000 \qquad p = 5 \qquad \max_{\alpha_i} = 1, \text{ for } i = 1, \ldots, p$$





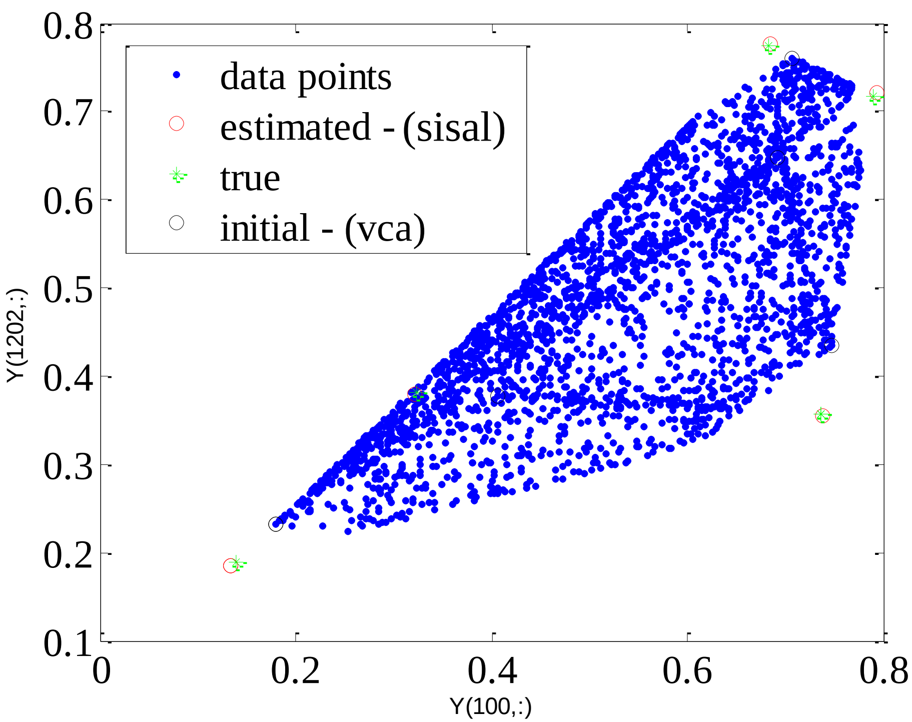
Time:

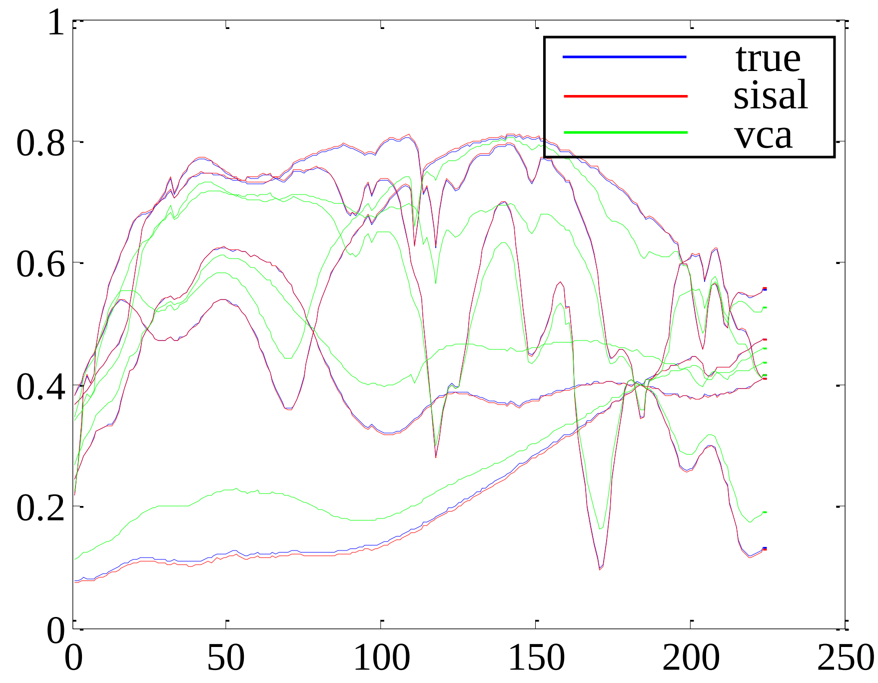
VCA → 0.5 sec

SISAL →2 sec

Example: Data set does not contain pure pixels

$$N = 5000 \qquad p = 5 \qquad \max_{\alpha_i} = 0.8, \text{ for } i = 1, \ldots, p$$

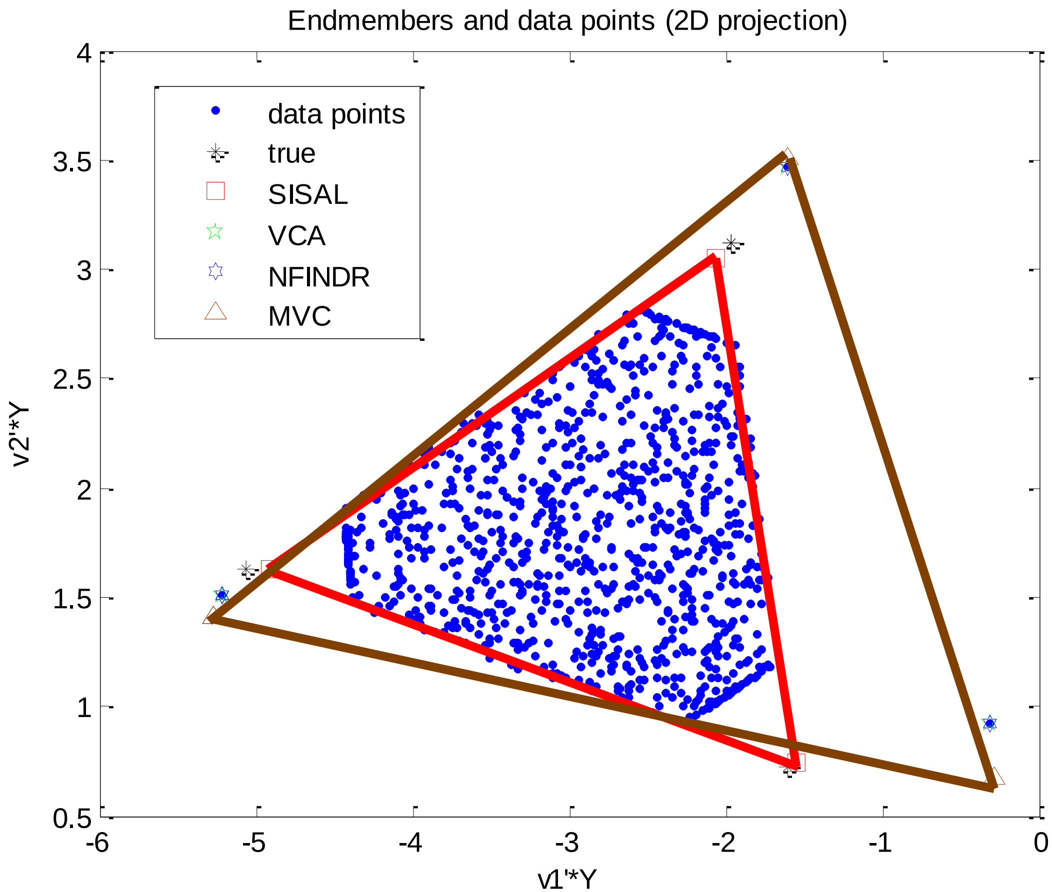




No pure pixels and outliers

$$N = 1000 \quad p = 3 \quad \max_{\alpha_i} = 0.8, \text{ for } i = 1, \ldots, p$$

$$\text{no. outliers} = 3$$



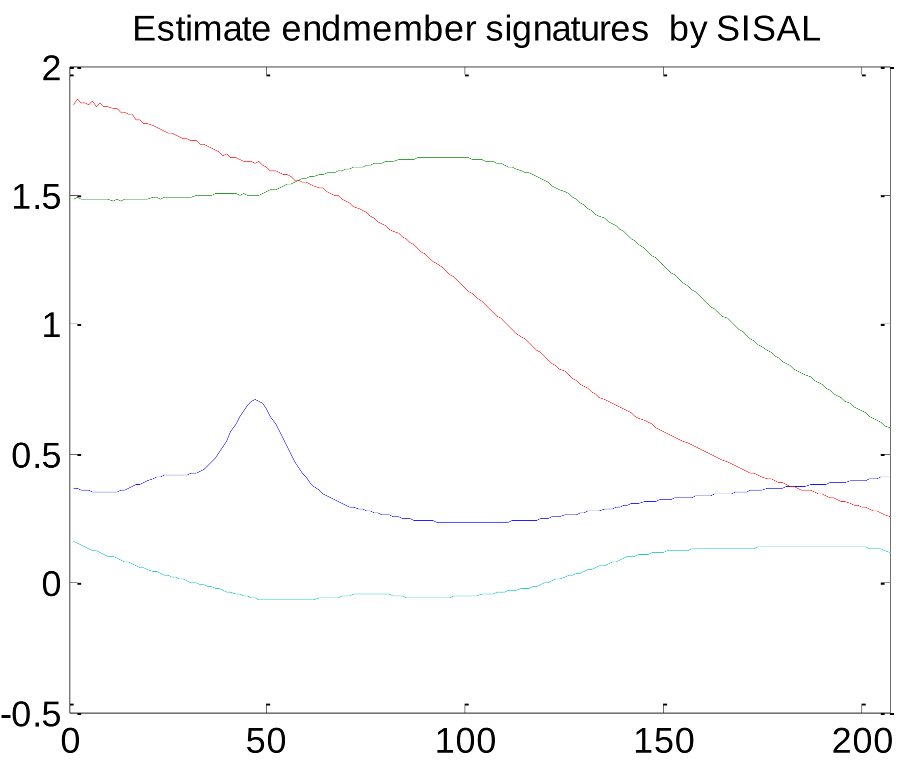
ERROR(mse):
- SISAL = 0.03
- VCA = 0.88
- NFINDR = 0.88
- MVC-NMF = 0.90

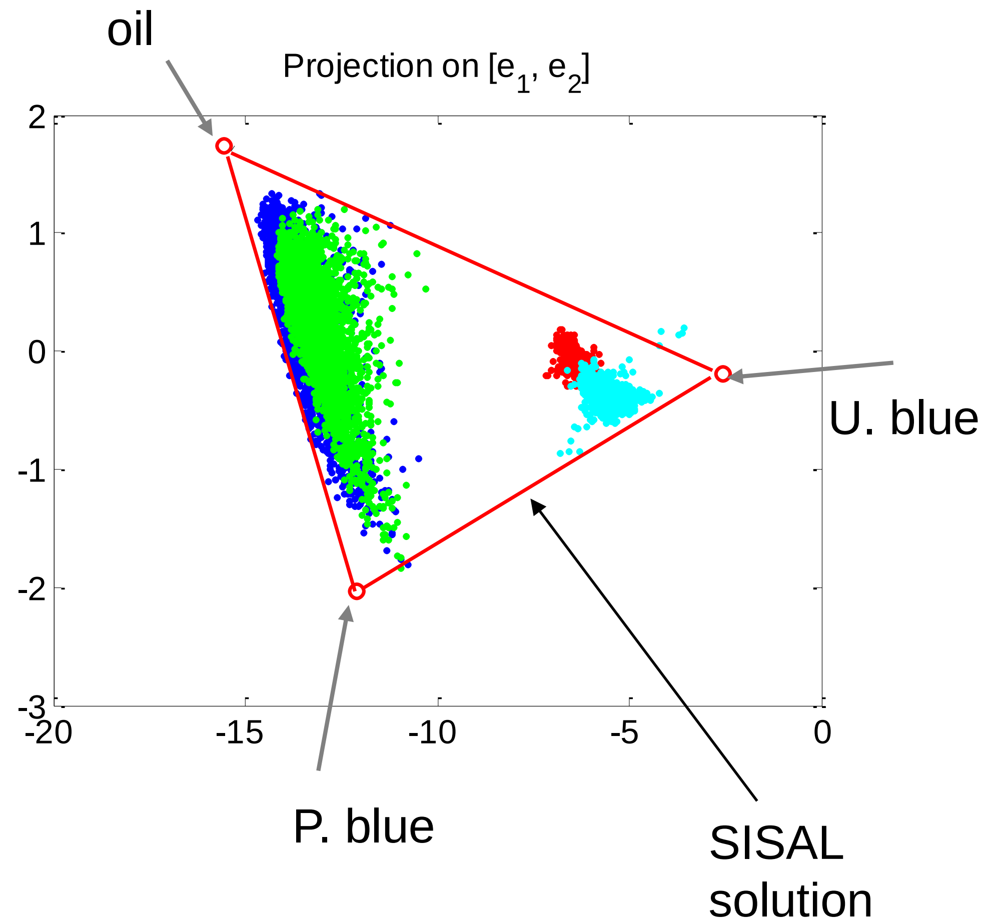
TIMES (sec):
- SISAL = 0.61,
- VCA = 0.20
- NFINDR = 0.25
- MVC-NMF = 25

Real data: ArtImageDataA (converted into absorbance)

Tiles 1,2 ('Prussian blue' + oil)

Tiles 15,16 ('Ultramarine blue' + oil)





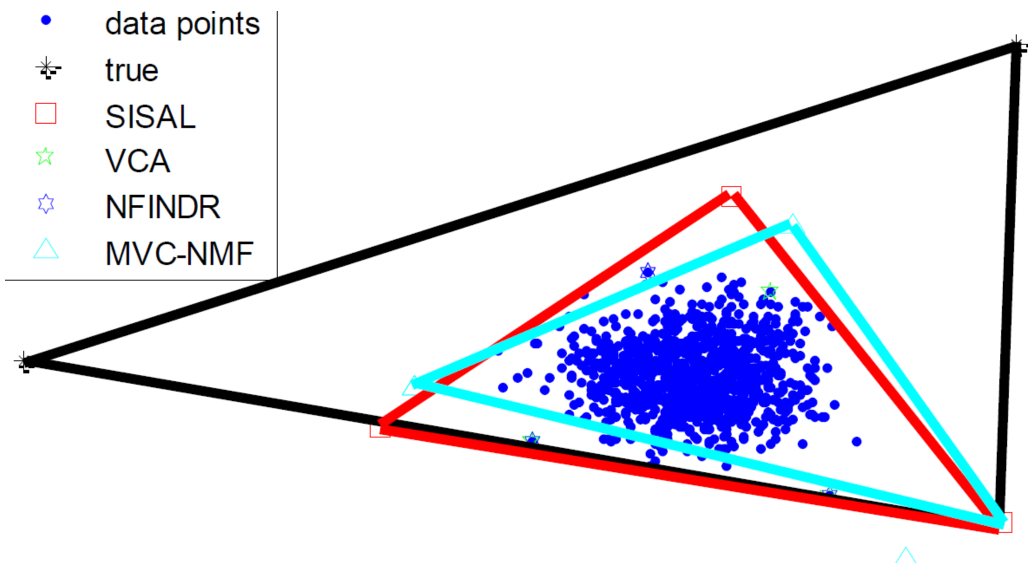
Geometrical Approaches: Limitations

- ❑ determined by a small number of pixels; some may be outliers

- ❑ MVS – Computationally heavy

- ❑ PPI, N-FINDR, VCA, SGA, AMEE, SVMAX, AVMAX depend on the existence of pure pixels in the data

- ❑ Do not work in highly mixed scenarios



Statistical approaches

- observation model $\mathbf{Y} = \mathbf{M}\mathbf{S} + \mathbf{N}$
- prior (Bayesian framework) $p_S(\mathbf{S})p_M(\mathbf{M})$
- posterior density

$$p_{M,S|Y}(\mathbf{M},\mathbf{S}|\mathbf{Y}) = p_{Y|M,S}(\mathbf{Y}|\mathbf{M},\mathbf{S})p_M(\mathbf{M})p_S(\mathbf{S})/p_Y(\mathbf{Y})$$

- inference

$$\begin{aligned}(\widehat{\mathbf{M}},\widehat{\mathbf{S}})_{\mathrm{MAP}} &\equiv \arg\max_{\mathbf{M},\mathbf{S}} p_{M,S|Y}(\mathbf{M},\mathbf{S}|\mathbf{Y}) \\ &= \arg\min -\log p_{Y|M,S}(\mathbf{Y}|\mathbf{M},\mathbf{S}) - \log p_M(\mathbf{M}) - \log p_S(\mathbf{S})\end{aligned}$$

$$\begin{aligned}\widehat{\mathbf{M}}_{\mathrm{MMSE}} &\equiv \mathbb{E}[\mathbf{M}|\mathbf{Y}] = \int \mathbf{M} p_{M|Y}(\mathbf{M}|\mathbf{Y})d\mathbf{M} \\ \widehat{\mathbf{S}}_{\mathrm{MMSE}} &\equiv \mathbb{E}[\mathbf{S}|\mathbf{Y}] = \int \mathbf{S} p_{S|Y}(\mathbf{S}|\mathbf{Y})d\mathbf{S}.\end{aligned}$$

Spectral linear unmixing and ICA/IFA

- Formally, SLU is a linear source separation problem

- Independent Component Analysis (ICA) come to mind

- ICA
  - *Fastica*, [Hyvarinen & Oja, 2000]
  - *Jade*, [Cardoso, 1997]
  - *Bell and Sejnowski*, [Bell and Sejnowski , 1995]

IFA
  - *IFA*, [Moulines *et al.*, 1997], [Attias, 1999]

Statistical approaches: ICA

$$\mathbf{y} = \mathbf{M}\boldsymbol{\alpha} + \mathbf{n}$$

Assumptions

1. Fractional abundances (sources) are independent

$$p_\alpha(\alpha) = p_1(\alpha_1)p_2(\alpha_2)\ldots p_k(\alpha_p)$$

2. Non-Gaussian sources

Endmembers compete for the same area $\sum_{j=1}^{k} \alpha_j = 1$

⇒ Sources are Dependent

⇒ ICA does not apply [Nascimento, B-D, 2005]

Representative Bayesian approaches

[Parra *et al*., 00] $(\mathbf{S}-$ u.d. on the simplex, $\mathbf{M}-$ AR model, MAP$)$

[Moussaoui *et al*., 06,a,b], [Dobigeon *et al*., 09,a,b], [Dobigeon *et al*., 09,b], [Arngreen, 11]

- data term associated with the LMM
- $\mathbf{S}-$Uniform on the simplex
- conjugate prior distributions for some unknown parameters
- Infer MMSE estimates by Markov chain Monte Carlo algorithms

DECA - [Nascimento, B-D 09, 14]

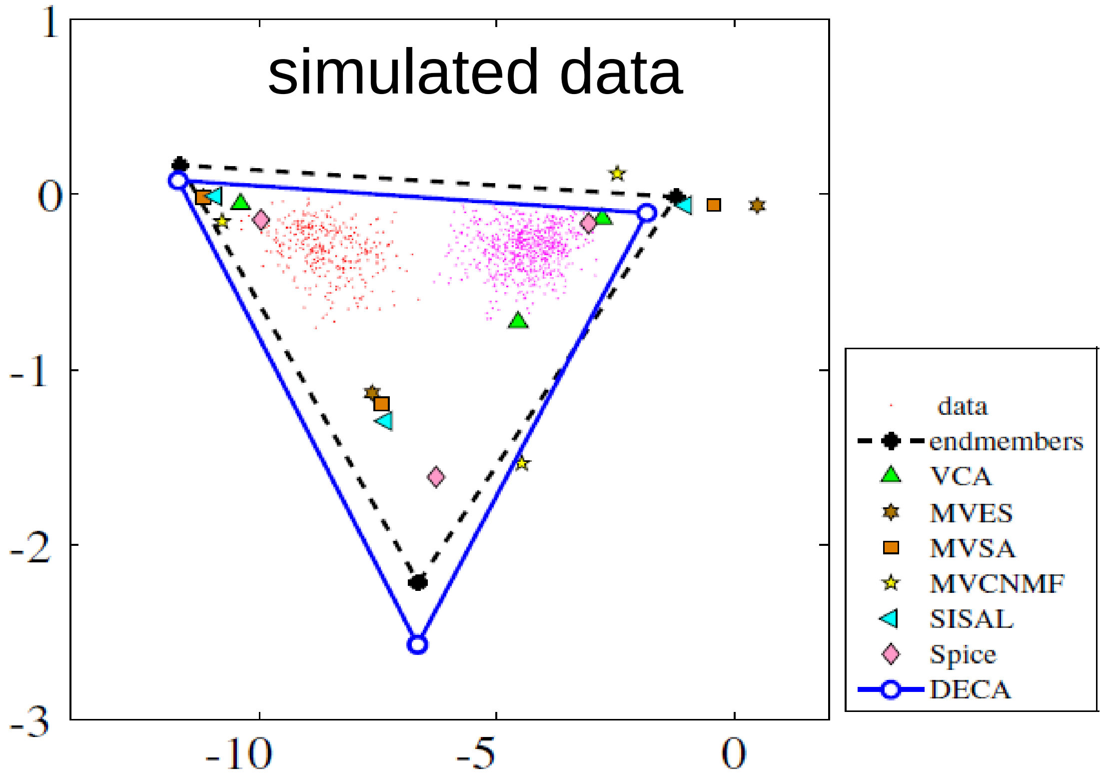
- data term associated with a noiseless LMM
- $\boxed{\mathbf{S}-\text{ Dirichlet mixture model}}$
- MDL based inference of the number of Dirichlet modes
- MAP inference (GEM - algorithm)

DECA – Dependent component analysis

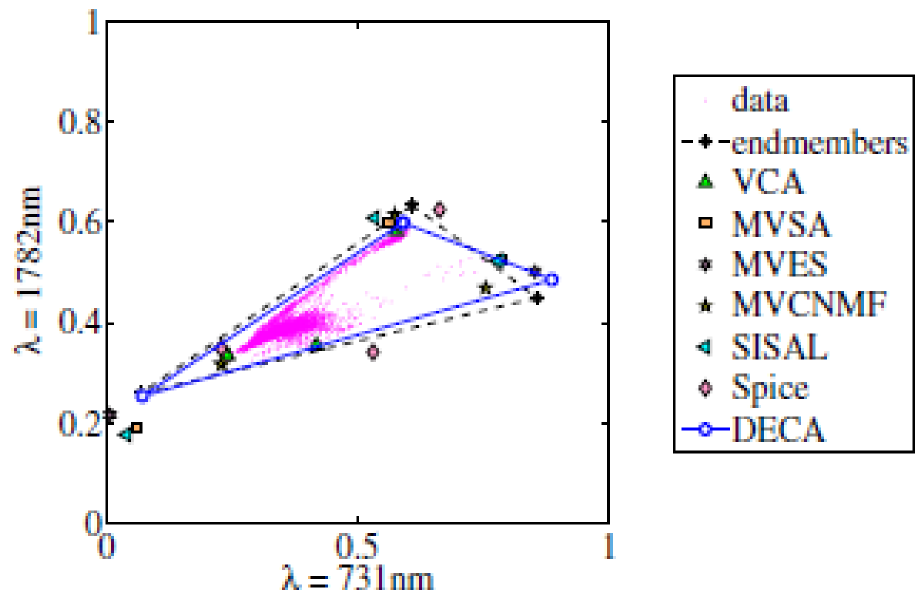
$$p_{Y|M}(\mathbf{Y}|\mathbf{M},\boldsymbol{\theta}) = \left(\prod_{i=1}^{N} p_\alpha(\mathbf{M}^{-1}\mathbf{y}_i,\boldsymbol{\theta})\right)|\det(\mathbf{M}^{-1})|^N$$

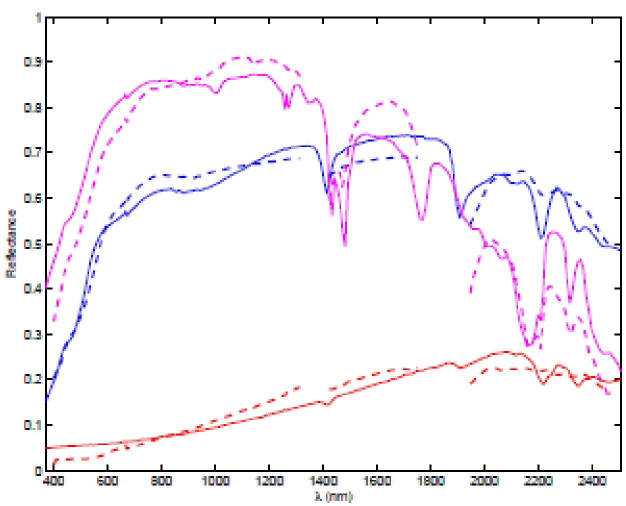
parameters of the Dirichlet mixture

minimum volume term



DECA – Results on Cuprite





[Mittelman, Dobigeon, Hero, 12]

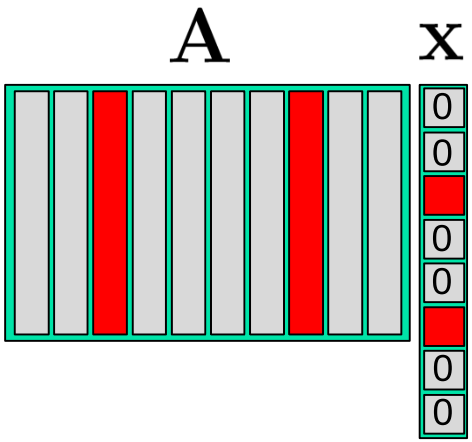
- data term associated with the LMM
- $\mathbf{S}-$ Dirichlet mixture model
- Latent label process enforcing adjacent pixels to have the same label
- spatial prior: tree-structured sticky hierarchical Dirichlet process (SHDP)
- MMSE inference by MCMC
- Model order inference (number of endmembers)

Sparse regression-based SLU

- Spectral vectors can be expressed as linear combinations of a few pure spectral signatures obtained from a (potentially very large) spectral library

[Iordache, B-D, Plaza, 11, 12]

$$\mathbf{y} = \sum_{i \in S} \mathbf{a}_i \mathbf{x}_i = \mathbf{A}\mathbf{x}$$



- Unmixing: given $\mathbf{y}$ and $\mathbf{A}$, find the sparsest solution of

$$\mathbf{y} = \mathbf{A}\mathbf{x}$$

- Advantage: sidesteps endmember estimation

- Disadvantage: **Combinatorial problem !!!**

Convex approximations to P0

CBPDN – Constrained basis pursuit denoising

$$\min_{\mathbf{x}} \|\mathbf{x}\|_1 \ \text{ subject to } \ \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2 \le \delta, \ \ \mathbf{x} \ge \mathbf{0},$$

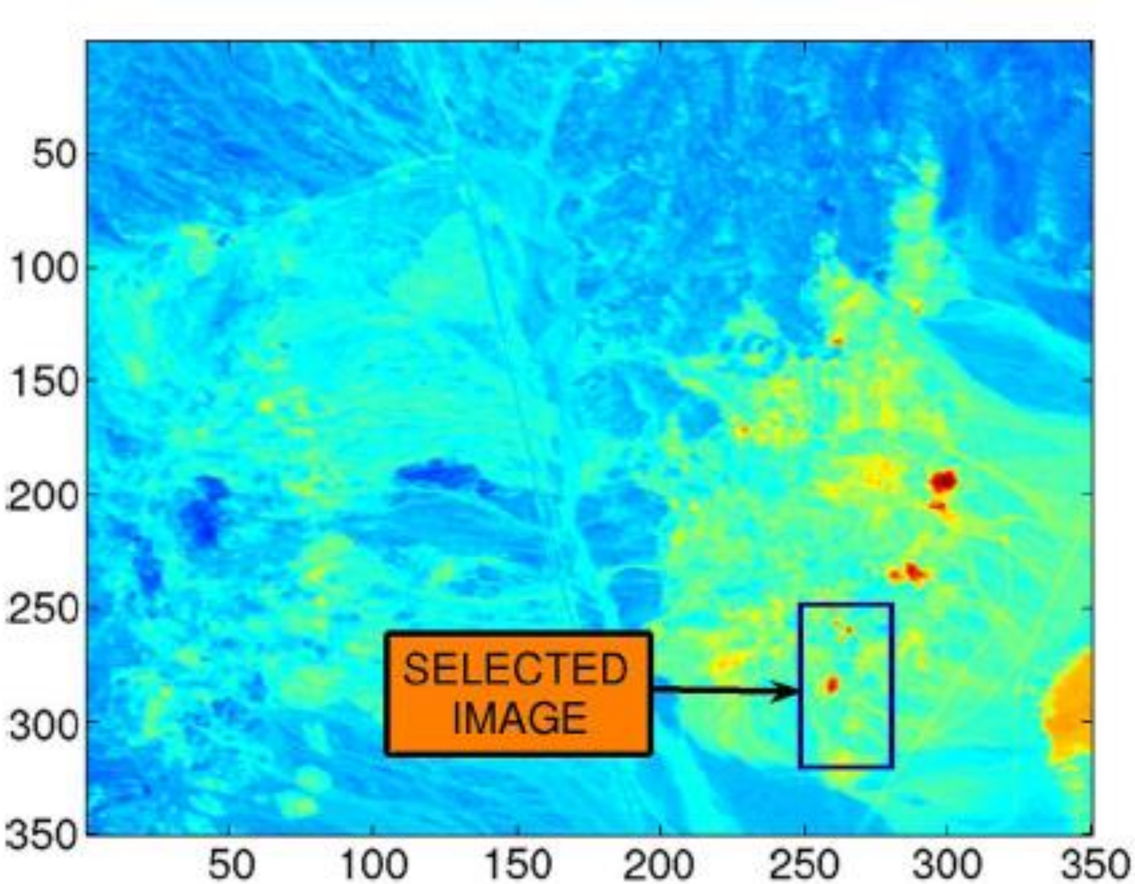
Equivalent problem

$$\min_{\mathbf{x}} (1/2)\|\mathbf{y} - \mathbf{A}\mathbf{x}\|^2 + \lambda\|\mathbf{x}\|_1, \quad \mathbf{x} \ge 0$$

Striking result: In given circumstances, related with the coherence among the columns of matrix **A**, BP(DN) yields the sparsest solution ([Donoho 06], [Candès et al. 06]).

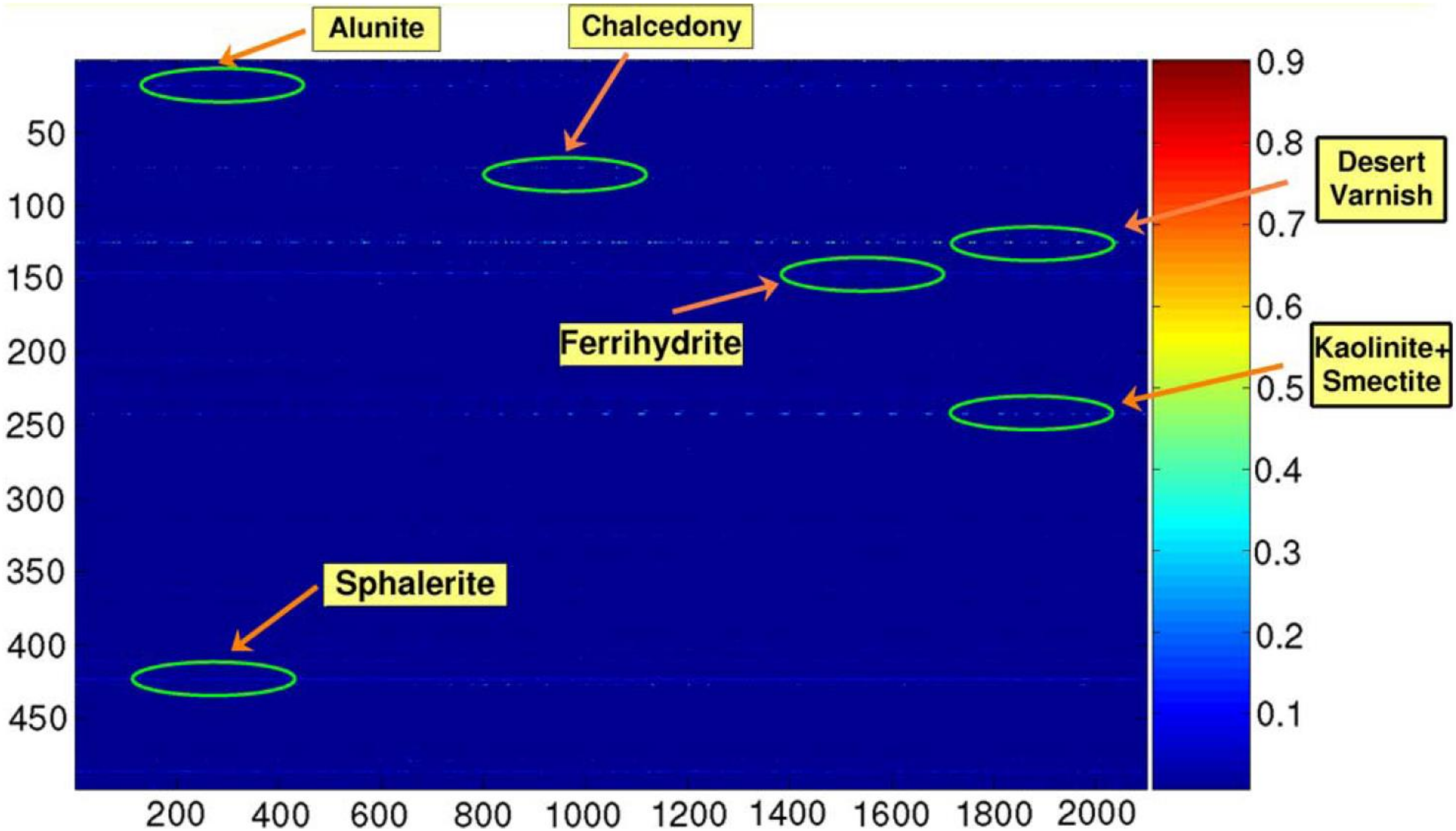
Efficient solvers for CBPDN: SUNSAL, CSUNSAL [B-D, Figueiredo, 10]

Real data – AVIRIS Cuprite



[Iordache, B-D, Plaza, 11, 12]

Real data – AVIRIS Cuprite



[Iordache, B-D, Plaza, 11, 12]

Sparse reconstruction of hyperspectral data: Summary

Bad news: Hyperspectral libraries have poor RI constants

Good news: Hyperspectral mixtures are highly sparse, very often $p \cdot 5$

Surprising fact: Convex programs (BP, BPDN, LASSO, …) yield much better empirical performance than non-convex state-of-the-art competitors

Directions to improve hyperspectral sparse reconstruction

- Structured sparsity + subspace structure
  (pixels in a give data set share the same support)

- Spatial contextual information
  (pixels belong to an image)

Constrained total variation sparse regression (CTVSR)

$$\min_{\mathbf{X}} (1/2)\|\mathbf{AX}-\mathbf{Y}\|_F^2 + \lambda_1\|\mathbf{X}\|_1 + \lambda_2\phi_{TV}(\mathbf{X})$$

[Iordache, B-D, Plaza, 11]

$$\text{subject to: } \mathbf{X} \geq \mathbf{0}$$

Total Variation of
i-th band

$$\phi_{TV}(\mathbf{X}) := \sum_{i=i}^{n} \|\mathbf{Lx}^i\|_1 = \sum_{i=1}^{n}\sum_{k=1}^{N}\sqrt{([\mathbf{D}_h\mathbf{x}^i]_k)^2 + ([\mathbf{D}_v\mathbf{x}^i]_k)^2}$$

Related work [Zhao, Wang, Huang, Ng, Plemmons, 12]

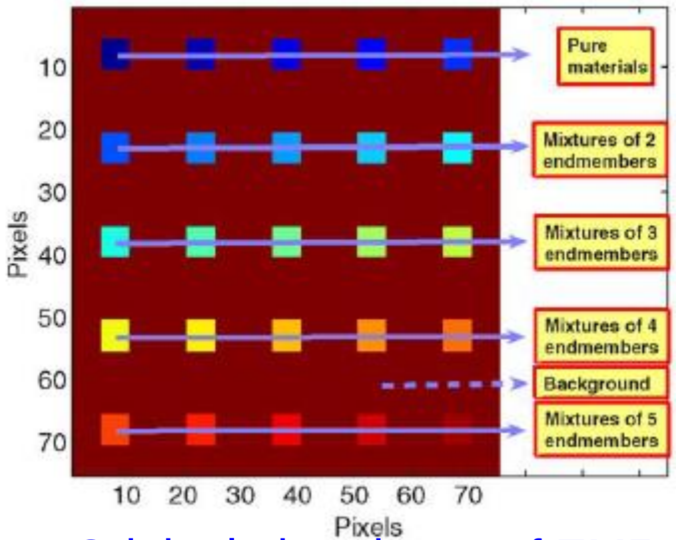
Other Regularizers:

- vector total variation (VTV)! promotes piecewise smoo vectors [Bresson, Chan, 02], [Goldluecke et al., 12], [Yuan, Zhang, Shen, 12]
- convex generalizations of Total Variation based on the Structure Tensor [Lefkimmiatis et al., 13]
- sparse representation (2D, 3D) in the wavelet domain

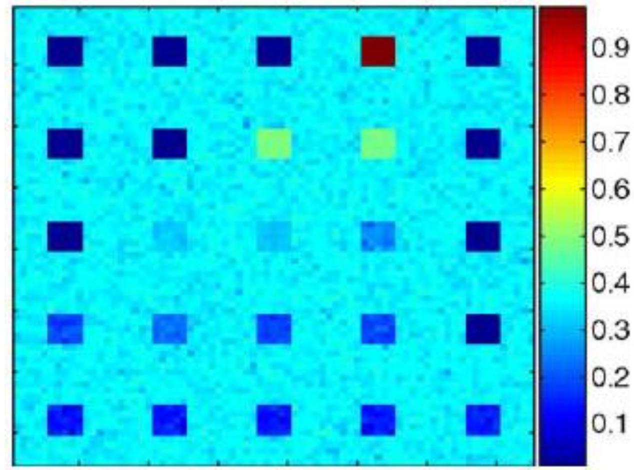
Ilustrative examples with simulated data : SUnSAL-TV

$\mathbf{A} \in \mathbb{R}^{224 \times 240}$ (from USGS library) $\qquad (m = 224, N = 75 \times 75, k = 5)$

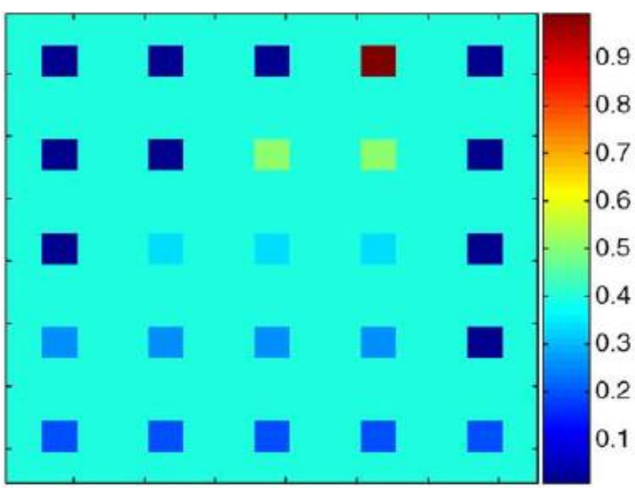
Original data cube



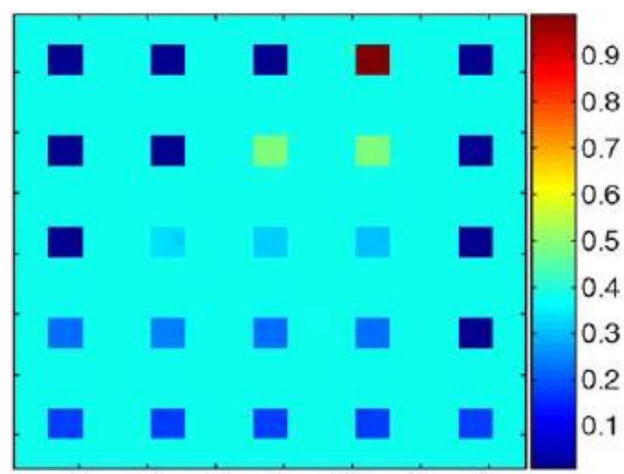
SUnSAL estimate



Original abundance of EM5



SUnSAL-TV estimate



Constrained colaborative sparse regression (CCSR)

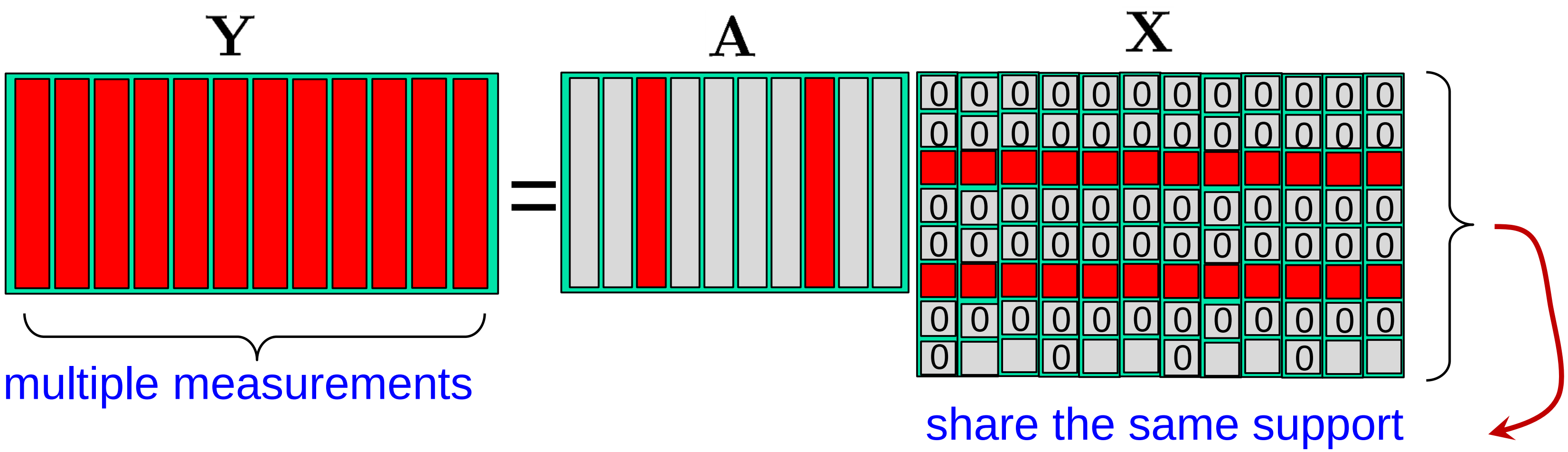
$$\min_{\mathbf{X}} (1/2)\|\mathbf{AX} - \mathbf{Y}\|_F^2 + \lambda \|\mathbf{X}\|_{2,1}$$

$$\|\mathbf{X}\|_{2,1} := \sum_{i=1}^{n} \|\mathbf{x}^i\|_2$$

$$\text{subject to: } \mathbf{X} \geq \mathbf{0}, \quad \mathbf{1}_n^T \mathbf{X} = \mathbf{1}_N^T$$

[Iordache, B-D, Plaza, 11, 12]

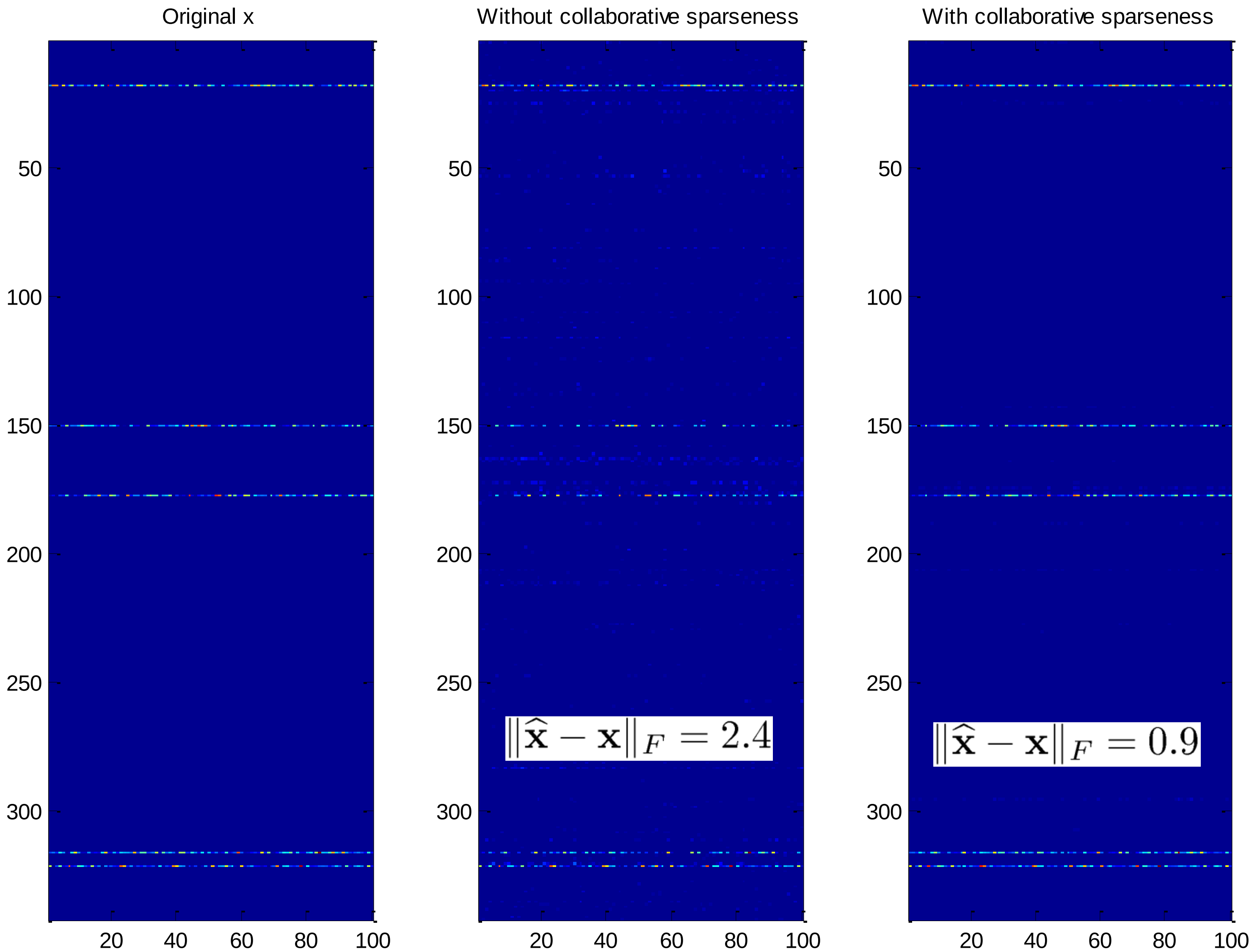
[Turlach, Venables, Wright, 2004]



Theoretical guaranties (superiority of multichanel) : the probability of recovery failure decays exponentially in the number of channels. [Eldar, Rauhut, 11]

Ilustrative examples with simulated data : CSUnSAL

$\mathbf{A} \in \mathbb{R}^{224\times 350}$ (from USGS library) $\mathbf{x} \in \mathbb{R}^{350\times 100}$ (sparsity k = 5)



$\mathrm{SNR} = 35\mathrm{dB}$

$\mathrm{time} = 10$ sec

MUSIC – Colaborative SR algorithm

MUSIC-CSR algorithm [Iordahe, B-D, Plaza, 2013]

1) Estimate the signal subspace $\mathrm{span}\{\mathbf{A}_S\}$using, e.g. the HySime algorithm.

2) Compute $\varepsilon_i = \dfrac{\|\mathbf{P}_y^\perp \mathbf{a}_i\|}{\|\mathbf{a}_i\|}$, for $i = 1, \ldots, m$ and define

the index set $S = [i \,:\, \varepsilon_i \le \delta, i = 1, \ldots, m]$

3) Solve the colaborative sparse regression optimization

$$\min_{\mathbf{X}} (1/2)\|\mathbf{Y} - \mathbf{A}_\mathbf{S}\mathbf{X}\|^2 + \lambda\|\mathbf{X}\|_{2,1}, \quad \mathbf{X} \ge 0$$

[B-D, Figueiredo, 2012]

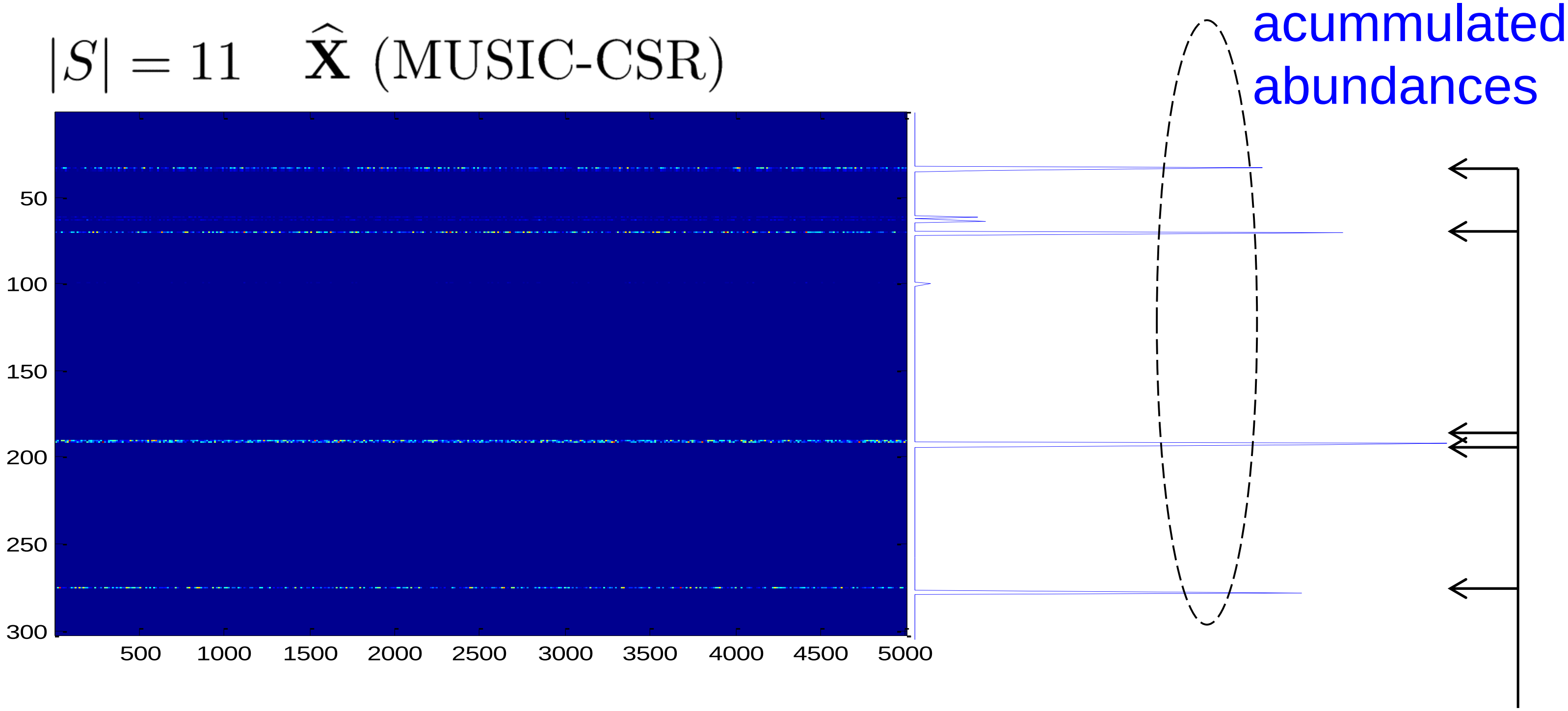
Related work: CS-MUSIC [Kim, Lee, Ye, 2012]

(N < k and iid noise)

MUSIC – CSR results

A – USGS $(\geq 3^\circ)$, Gaussian shaped noise, SNR = 25 dB, k = 5, m = 300,

$|S| = 11 \quad \widehat{\mathbf{X}}$ (MUSIC-CSR)



MUSIC-CSR
- SNR = 11.7 dB
- computation time ' 10 sec

CSR
- SNR = 0 dB
- computation time ' 600 sec

Brief Concluding remarks

- HU is a hard inverse problem (noise, bad-conditioned direct operators, nonlinear mixing phenomena)
- HU calls for sophisticated math tools and framework (statistical inference, optimization, machine learning)
- The research efforts devoted to non-linear mixing models are increasing

Linear mixing

- Apply geometrical approaches when there are data vectors near or over the simplex facets
- Apply statistical methods in highly mixed data sets
- Apply sparse regression methods, if there exits a spectral library for the problem in hand

Spectral nonlinear unmixing (SNLU). Just a few topics

- Detecting nonlinear mixtures in polynomial post-nonlinear mixing model, [Altmann, Dobigeon, Tourneret, 11,13]

$$\mathbf{y} = \mathbf{M}\boldsymbol{\alpha} + b(\mathbf{M}\boldsymbol{\alpha}) \odot (\mathbf{M}\boldsymbol{\alpha}) + \mathbf{n}$$

hypothesis test

- Bilinear unmixing model, [Fan, Hu, Miller, Li, 09], [Nascimento, B-D, 09], [Halimi, Altmann, Dobigeon, Tourneret, 11,11]

$$\mathbf{y} = \sum_{i=1}^{p} \alpha_i \mathbf{m}_i + \sum_{i,j=1}^{p} \alpha_{i,j} \mathbf{m}_i \odot \mathbf{m}_j + \mathbf{n}$$

- Kernel-based unmixing algorithms to specifically account for intimate mixtures [Broadwater, Chellappa, Burlina, 07], [Broadwater, Banerjee, 09,10, 11], [Chen, Richard, Ferrari, Honeine, 13]

N. Dobigeon, J.-Y. Tourneret, C. Richard, J. C. M. Bermudez, S. McLaughlin and A. O. Hero, "Nonlinear unmixing of hyperspectral images: models and algorithms," IEEE Signal Process. Magazine, vol. 31, no 1, pp. 82-94, 2014

R. Heylen, M.Parente, and P. Gader, "A review of nonlinear hyperspectral unmixing methods," Selected Topics in Applied Earth Observations and Remote Sensing, IEEE Journal of , vol.7, no.6, pp.1844-1868, 2014

Sparse regression-based SLU

$\mathbf{A} \in \mathbb{R}^{L \times m}$ (library, $p \ll L < m$, undetermined system)

Problem – P0

$$\min_{\mathbf{x}} \|\mathbf{x}\|_0 \text{ subject to } \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2 \leq \delta, \; \mathbf{x} \geq \mathbf{0},$$

Very difficult (NP-hard)

Approximations to P0**:**

OMP – orthogonal matching pursuit [Pati *et al.*, 2003]

BP – basis pursuit [Chen *et al.*, 2003]

BPDN – basis pursuit denoising

IHT (see [Blumensath, Davis, 11], [Kyrillidis, Cevher, 12])